# *Chapter 3. Renewal Processes*

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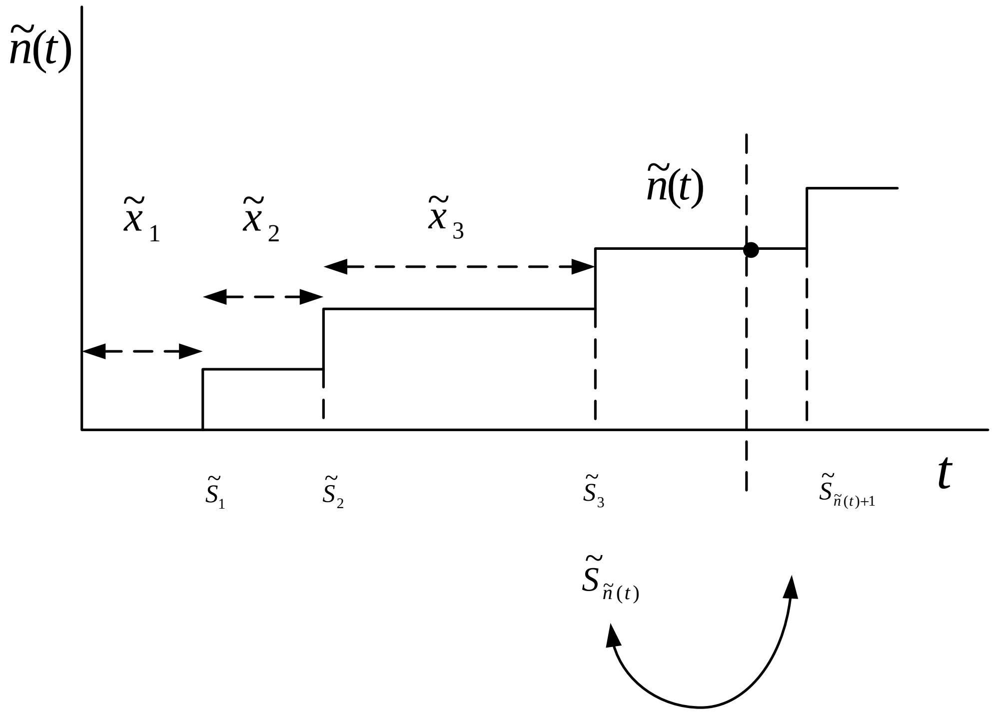
# Outline

- Distribution and Limiting Behavior of $\tilde{n}(t)$
  - Pmf of $\tilde{n}(t)$ : $P(\tilde{n}(t) = k) = ?$
  - Limiting time average : $\lim_{t\to\infty} \frac{\tilde{n}(t)}{t} = ?$ (Law of Large Numbers)
  - Limiting PDF of $\tilde{n}(t)$ (Central Limit Theorem)
- Renewal Function $E[\tilde{n}(t)]$, and its Asymptotic (Limiting) behavior
  - Renewal Equation
  - Wald's Theorem and Stopping time
  - Elementary Renewal Theorem
  - Blackwell's Theorem

# Outline

- Key Renewal Theorem and Applications
  - Definition of Regenerative Process
  - Renewal Theory
  - Key Renewal Theorem
  - Application 1: Residual Life, Age, and Total Life
  - Application 2: Alternating Renewal Process/Theory
  - Application 3: Mean Residual Life
- Renewal Reward Processes and Applications
  - Renewal Reward Process/Theory
  - Application 1: Alternating Renewal Process/Theory
  - Application 2: Time Average of Residual Life and Age
- More Notes on Regenerative Processes

# Distribution and Limiting Behavior of $\tilde{n}(t)$




$\{\tilde{x}_n, n = 1, 2, \ldots\} \sim F_{\tilde{x}}$; mean $\bar{X}$ $(0 < \bar{X} < \infty)$
$N = \{\tilde{n}(t), t \geq 0\}$ is called a renewal (counting) process

$$\begin{aligned} \tilde{n}(t) &= \sup\{n : \tilde{S}_n \leq t\} \quad (\because \text{There are always finite renewals} \\ &= \max\{n : \tilde{S}_n \leq t\} \quad \text{in a finite time } (i.e., \tilde{n}(t) < \infty)) \end{aligned}$$

# Distribution and Limiting Behavior of $\tilde{n}(t)$

$\boxed{\tilde{n}(t)}$

1. pmf of $\tilde{n}(t) \rightarrow$ closed-form
2. Limiting time average [Law of Large Numbers]:

$$\frac{\tilde{n}(t)}{t} \overset{w.p.1}{\longrightarrow} \frac{1}{\bar{\bar{X}}}, \; t \rightarrow \infty$$

3. Limiting time and ensemble average
   [Elementary Renewal Theorem]:

$$\frac{E[\tilde{n}(t)]}{t} \overset{w.p.1}{\longrightarrow} \frac{1}{\bar{\bar{X}}}, \; t \rightarrow \infty$$

   Items 2 and 3 $\rightarrow$ Ergodic Theory

# Distribution and Limiting Behavior of $\tilde{n}(t)$

4. Limiting ensemble average (focusing on arrivals in the vicinity of $t$ ) [Blackwell's Theorem]:

$$\frac{E[\tilde{n}(t+\delta) - \tilde{n}(t)]}{\delta} \overset{w.p.1}{\longrightarrow} \frac{1}{\bar{X}}\ ,\ t \to \infty$$

5. Limiting PDF of $\tilde{n}(t)$ [Central Limit Theorem]:

$$\lim_{t\to\infty} P\left[\frac{\tilde{n}(t) - t/\bar{X}}{\sigma\sqrt{t}(\bar{X})^{-3/2}} < y\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \sim Gaussian(\frac{t}{\bar{X}}, \sigma\sqrt{t}\cdot\bar{X}^{-\frac{3}{2}})$$

# pmf of $\tilde{n}(t)$

$$
\begin{aligned}
P[\tilde{n}(t) = n] &= P[\tilde{n}(t) \geq n] - P[\tilde{n}(t) \geq n+1] \\
&= P[\tilde{S}_n \leq t] - P[\tilde{S}_{n+1} \leq t] \\
& \qquad\qquad\qquad \because \tilde{x}_i \sim F, \\
& \qquad\qquad\qquad \therefore \sum \tilde{x}_i \sim F(t) \otimes F(t) \ldots \otimes F(t) \equiv F_n(t) \\
&= F_n(t) - F_{n+1}(t) \qquad\qquad n\text{-fold convolution of } F(t)
\end{aligned}
$$

# Limiting Time Average

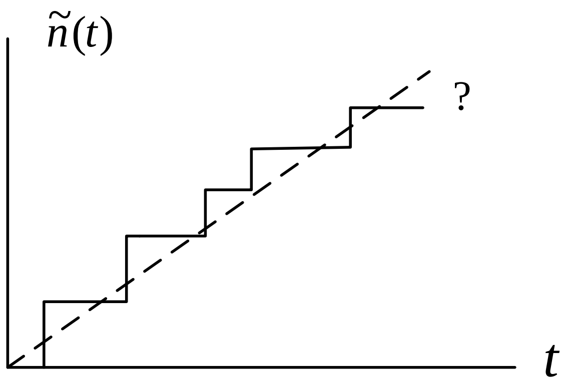
$\lim_{t\to\infty} \tilde{n}(t) = ?$

$$\because P\left[\lim_{t\to\infty} \tilde{n}(t) < \infty\right] = P[\tilde{n}(\infty) < \infty] = P[\tilde{x}_n = \infty \text{ for some } n]$$

$$= P\left[\bigcup_{n=1}^{\infty} (\tilde{x}_n = \infty)\right] = \sum_{n=1}^{\infty} P[\tilde{x}_n = \infty] = 0$$

$$\therefore \lim_{t\to\infty} \tilde{n}(t) = \tilde{n}(\infty) = \infty \quad w.p.1$$

Question: What is the rate at which $\tilde{n}(t)$ goes to $\infty$ ?




i.e. $\lim_{t\to\infty} \frac{\tilde{n}(t)}{t} = ?$

# Strong Law for Renewal Processes

**Theorem.** For a renewal process $N = \{\tilde{n}(t), t \geq 0\}$ with mean inter-renewal interval $\bar{X}$, then

$$\lim_{t \to \infty} \frac{\tilde{n}(t)}{t} = \frac{1}{\bar{X}},\ w.p.1$$

**Proof.**

# Central Limit Theorem for $\tilde{n}(t)$

**Theorem.** Assume that the inter-renewal intervals for a renewal process $N = \{\tilde{n}(t), t \geq 0\}$ have finite mean and variance $\bar{X}$, $\sigma^2$. Then,

$$\lim_{t\to\infty} P\left[\frac{\tilde{n}(t) - t/\bar{X}}{\sigma\sqrt{\dfrac{t}{\bar{X}^3}}} < y\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\dfrac{x^2}{2}}\,dx$$

**Proof.** (idea: $\tilde{n}(t) \rightarrow \tilde{S}_{\tilde{n}(t)} \rightarrow CLT$)

# Renewal Function $E[\tilde{n}(t)]$

Let $m(t) = E[\tilde{n}(t)]$, which is called "*renewal function*".

1. Relationship between $m(t)$ and $F_n$

$$m(t) = \sum_{n=1}^{\infty} F_n(t), \qquad \text{where } F_n \text{ is the } n\text{-fold convolution of } F$$

2. Relationship between $m(t)$ and $F$
   [Renewal Equation]

$$m(t) = F(t) + \int_0^t m(t-x)dF(x)$$

3. Relationship between $m(t)$ and $L_{\tilde{x}}(r)$ (Laplace Transform of $\tilde{x}$)

$$L_m(r) = \frac{L_{\tilde{x}}(r)}{r[1 - L_{\tilde{x}}(r)]}$$

# Renewal Function $E[\tilde{n}(t)]$

$\rightarrow$ [Wald's Equation]

4. Asymptotic behavior of $m(t)$ ($t \rightarrow \infty$, Limiting)
   $\rightarrow$ [Elementary Renewal Theorem]
   $\rightarrow$ [Blackwell's Theorem]

# Renewal Function $E[\tilde{n}(t)]$

1. $m(t) = E[\tilde{n}(t)] \overset{?}{\longleftrightarrow} F_n$ (i.e., PDF of $\tilde{S}_n$)

$$\text{Let } \tilde{n}(t) = \sum_{n=1}^{\infty} I_n, \text{ where } I_n = \begin{cases} 1, & n_{th} \text{ renewal occurs in } [0,t]; \\ 0, & \text{Otherwise;} \end{cases}$$

$$\begin{aligned} m(t) = E[\tilde{n}(t)] &= E\left[\sum_{n=1}^{\infty} I_n\right] \\ &= \sum_{n=1}^{\infty} E[I_n] \\ &= \sum_{n=1}^{\infty} P[\qquad\qquad\qquad\qquad\qquad\qquad] \\ &= \sum_{n=1}^{\infty} P[\qquad\qquad\qquad] \end{aligned}$$

## Renewal Function $E[\tilde{n}(t)]$

$$\therefore\ m(t) = \sum_{n=1}^{\infty} F_n(t)$$

or $$m(t) = \sum_{n=1}^{\infty} P[\tilde{n}(t) \geq n] = \sum_{n=1}^{\infty} P[\tilde{S}_n \leq t] = \sum_{n=1}^{\infty} F_n(t)$$

.................................................................

As $t \to \infty,\ n \to \infty$, finding $F_n$ is far too complicated

$\Rightarrow$ find another way of solving $m(t)$ in terms of $F_{\tilde{x}}(t)$

# Renewal Function $E[\tilde{n}(t)]$

2. $m(t) \stackrel{?}{\longleftrightarrow} F_{\tilde{x}}(t)$ (i.e., PDF of $\tilde{x}$)

$$\because \quad \tilde{S}_n = \tilde{S}_{n-1} + \tilde{x}_n, \text{ for all } n \geq 1, \text{ and } \tilde{S}_{n-1} \text{ and } \tilde{x}_n \text{ are independent,}$$

$$\therefore \quad P[\tilde{S}_n \leq t] = \int_0^t P[\tilde{S}_{n-1} \leq t - x] dF_{\tilde{x}}(x), \text{ for } n \geq 2$$

$$\text{for } n = 1, \tilde{x}_1 = \tilde{S}_1, P[\tilde{S}_1 \leq t] = F_{\tilde{x}}(x)$$

$$\therefore \quad m(t) = \sum_{n=1}^{\infty} P[\tilde{S}_n \leq t] = F_{\tilde{x}}(t) + \int_0^t \sum_{n=2}^{\infty} P[\tilde{S}_{n-1} \leq t - x] dF_{\tilde{x}}(x)$$

$$m(t) = F_{\tilde{x}}(t) + \int_0^t m(t-x) \cdot dF_{\tilde{x}}(x) \quad \Rightarrow \text{Renewal Equation}$$

# Renewal Function $E[\tilde{n}(t)]$

3. $L_m(r) \overset{?}{\longleftrightarrow} L_{\tilde{x}}(r)$ (Laplace Transform of $\tilde{x}$)
   (Laplace Transform of $m(t) = L_m(r)$)

   Answer:

$$L_m(r) = \frac{L_{\tilde{x}}(r)}{r[1 - L_{\tilde{x}}(r)]}$$

   <Homework> Prove it.

4. Asymptotic behavior of $m(t)$:

$$\lim_{t\to\infty} \frac{m(t)}{t} = \lim_{t\to\infty} \frac{E[\tilde{n}(t)]}{t} = ?$$

# Stopping Time (Rule)

**Definition.** $\tilde{N}$, an integer-valued r.v., is said to be a "stopping time" for a set of independent random variables $\tilde{x}_1, \tilde{x}_2, \ldots$ if event $\{\tilde{N} = n\}$ is independent of $\tilde{x}_{n+1}, \tilde{x}_{n+2}, \ldots$

**Example 1.**

- Let $\tilde{x}_1, \tilde{x}_2, \ldots$ be independent random variables,
- $P[\tilde{x}_n = 0] = P[\tilde{x}_n = 1] = 1/2, \quad n = 1, 2, \ldots$
- if $\tilde{N} = \min\{n : \tilde{x}_1 + \ldots + \tilde{x}_n = 10\}$

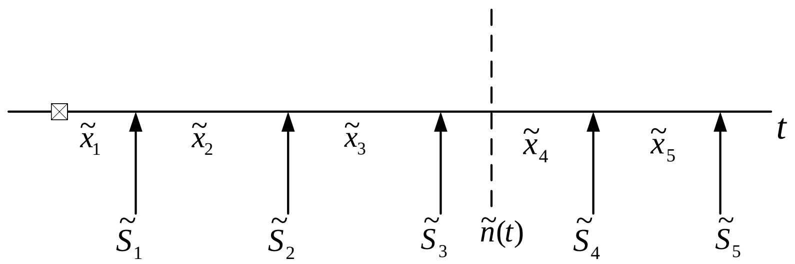
$\rightarrow$ Is $\tilde{N}$ a stopping time for $\tilde{x}_1, \tilde{x}_2, \ldots$?

**Answer:** .

# Stopping Time (Rule)

**Example 2.**

- $\tilde{n}(t)$, $X = \{\tilde{x}_n, n = 1, 2, 3, \ldots\}$,
- $S = \{\tilde{S}_n, n = 0, 1, 2, 3, \ldots\}$,
- $\tilde{S}_n = \tilde{S}_{n-1} + \tilde{x}_n$




$\rightarrow$ Is $\tilde{n}(t)$ the stopping time of $X = \{\tilde{x}_n, n = 1, 2, \ldots\}$?

**Answer:**

# Stopping Time (Rule)

**Example 3.** Is $\tilde{n}(t) + 1$ the stopping time for $\{\tilde{x}_n\}$?

**Answer:**

# Stopping Time - from $\tilde{I}_n$

**Definition.** $\tilde{N}$, an integer-valued r.v. is said to be a stopping time for a set of independent random variables $\{\tilde{x}_n, n \geq 1\}$, if for each $n > 1$, $\tilde{I}_n$, conditional on $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}$, is independent of $\{\tilde{x}_k, k \geq n\}$

**Define.** $\tilde{I}_n$ - a decision rule for stopping time $\tilde{N}, n \geq 1$

$$\tilde{I}_n = \begin{cases} 1, & \text{if the } n_{th} \text{ observation is to be made;} \\ 0, & \text{Otherwise} \end{cases}$$

1. $\because \tilde{N}$ is the stopping time
   $\therefore \tilde{I}_n$ depends on $\tilde{x}_1, \ldots, \tilde{x}_{n-1}$ but not $\tilde{x}_n, \tilde{x}_{n+1}, \ldots$
2. $\tilde{I}_n$ is also an indicator function of event $\{\tilde{N} \geq n\}$,
   i.e., $\tilde{I}_n = \begin{cases} 1, & \text{if } \tilde{N} \geq n; \\ 0, & \text{Otherwise;} \end{cases}$

# Stopping Time - from $\tilde{I}_n$

Because

- If $\tilde{N} \geq n$, then $n_{th}$ observation must be made;
- Since $\tilde{N} \geq n$ implies $\tilde{N} \geq n-1$ and happily, $\tilde{I}_n = 1$ implies $\tilde{I}_{n-1} = 1$

$\therefore$ Stopping time

$$\begin{cases} \{\tilde{N} = n\}, \text{ is} \\ \qquad \underline{\text{or}} \\ \tilde{I}_n \text{ is} \end{cases}$$

# Wald's Equation

**Theorem.** If $\{\tilde{x}_n, n \geq 1\}$ are i.i.d. random variables with finite mean $E[\tilde{x}]$, and if $\tilde{N}$ is the stopping time for $\{\tilde{x}_n, n \geq 1\}$, such that $E[\tilde{N}] < \infty$. Then,

$$E\left[\sum_{n=1}^{\tilde{N}} \tilde{x}_n\right] = E[\tilde{N}] \cdot E[\tilde{x}]$$

**Proof.**

# Wald's Equation

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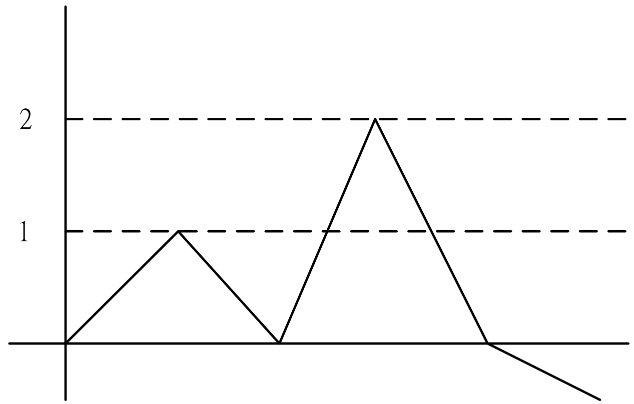
For Wald's Theorem to be applied, other than $\{\tilde{x}_i, i \geq 1\}$

1. $\tilde{N}$ must be a stopping time; and
2. $E[\tilde{N}] < \infty$

# Wald's Equation

**Example.** (Example 3.2.3 – Simple Random Walk, [Kao])




$\{\tilde{x}_i\}$ i.i.d. with:

$$P(\tilde{x} = 1) = p$$
$$P(\tilde{x} = -1) = 1 - p = q$$
$$\tilde{S}_n = \sum^n \tilde{x}_k$$

# Wald's Equation

- Let $\tilde{N} = \min\{n | \tilde{S}_n = 1\}$

# Wald's Equation

- Let $\tilde{M} = \min\{n|\tilde{S}_n = 1\} - 1$

# Corollary

Before proving $\lim_{t\to\infty} \frac{m(t)}{t} \to \frac{1}{\bar{X}}$,

**Corollary.** If $\bar{X} < \infty$, then

$$E[\tilde{S}_{\tilde{n}(t)+1}] = \bar{X}[m(t) + 1]$$

**Proof.**

# The Elementary Renewal Theorem

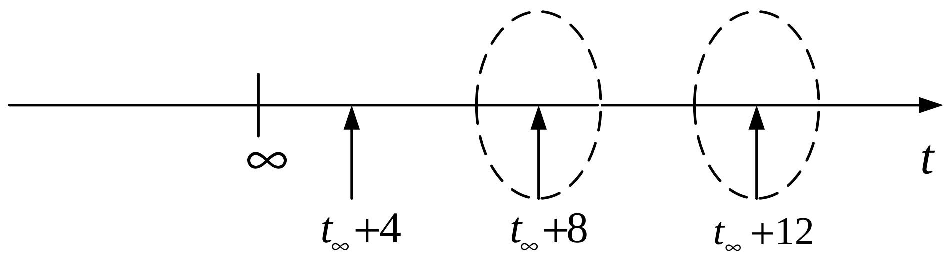
**Theorem.**

$$\frac{m(t)}{t} \to \frac{1}{\bar{X}} \quad \text{as } t \to \infty$$

**Proof.**

# Blackwell's Theorem

- **Ensemble Average.**
  - to determine the expected renewal rate in the limit of large $t$, without averaging from $0 \to t$ (time average)
- **Question.**
  - are there some values of $t$ at which renewals are more likely than others for large $t$ ?




  - **An example.** If each inter-renewal interval $\{\tilde{x}_i, i = 1, 2, \ldots\}$ takes on integer number of time units, e.g., $0, 4, 8, 12, \ldots$, then expected rate of renewals is zero at other times. Such random variable is said to be "*lattice*".

# Blackwell's Theorem

– **Definitions.**

  * A nonnegative random variable $\tilde{x}$ is said to be *lattice* if there exists $d \geq 0$ such that

$$\sum_{n=0}^{\infty} P[\tilde{x} = nd] = 1$$

  * That is, $\tilde{x}$ is lattice if it only takes on integral multiples of some nonnegative number $d$. The largest $d$ having this property is said to be the *period* of $\tilde{x}$. If $\tilde{x}$ is lattice and $F$ is the distribution function of $\tilde{x}$, then we say that $F$ is *lattice*.

- **Answer.**

  – Inter-renewal interval random variables are not lattice
  $\Rightarrow$ uniform expected rate of renewals in the limit of large $t$.
  (*Blackwell's Theorem*)

# Blackwell's Theorem

**Theorem.** If, for $\{\tilde{x}_i, i \geq 1\}$, which are not lattice, then, for any $\delta > 0$,

$$\lim_{t\to\infty}[m(t+\delta) - m(t)] = \frac{\delta}{\bar{\bar{X}}}$$

If the inter-renewal distribution is lattice with period $d$, then for any integer $n \geq 1$,

$$\lim_{n\to\infty} m(nd) = \frac{d}{\bar{\bar{X}}} \qquad (\text{or } \lim_{t\to\infty}[m(t+nd) - m(t)] = \frac{nd}{\bar{\bar{X}}})$$

**Proof.** (omitted)

# Blackwell's Theorem

For non-lattice inter-renewal process $\{\tilde{x}_i, i \geq 1\}$,

1. $\because \tilde{x}_i > 0 \Rightarrow$ No multiple renewals (single arrival)
2. From Blackwell's Theorem, the probability of a renewal in a small interval $(t, t+\delta]$ tends to $\delta/\bar{X} + o(\delta)$ as $t \to \infty$,

   $\therefore$ Limiting distribution of renewals in $(t, t+\delta]$ satisfies

$$\lim_{t\to\infty} P[\tilde{n}(t+\delta) - \tilde{n}(t) = 1] = \frac{\delta}{\bar{X}} + o(\delta)$$

$$\lim_{t\to\infty} P[\tilde{n}(t+\delta) - \tilde{n}(t) = 0] = 1 - \frac{\delta}{\bar{X}} + o(\delta)$$

$$\lim_{t\to\infty} P[\tilde{n}(t+\delta) - \tilde{n}(t) \geq 2] = o(\delta)$$

# Blackwell's Theorem

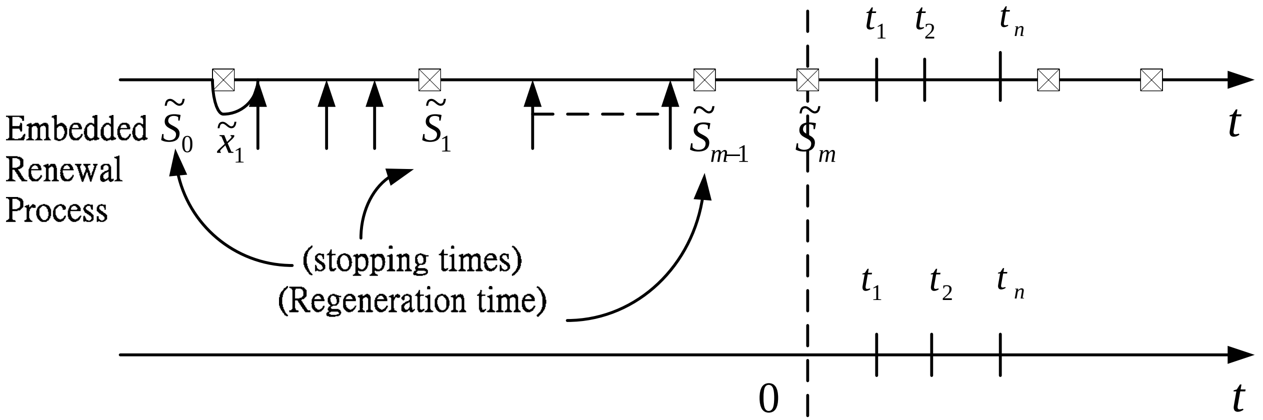
$\Rightarrow$

| | single arrival | Stationary Increment | Independent Increment |
|---|---|---|---|
| Poisson | | | |
| Renewal Process (Non-lattice) | | | |

# Regenerative Process

Regenerative Process

- $Z = \{\tilde{Z}_t, t \geq 0\}$; $\quad S = \{\tilde{S}_n, n \geq 0\}$ is a renewal process;




- $Z$ is said to be a regenerative process if

$$E[f(\tilde{Z}_{\tilde{S}_m+t_1}, \tilde{Z}_{\tilde{S}_m+t_2}, \ldots, \tilde{Z}_{\tilde{S}_m+t_n}) | \tilde{Z}_u; u \leq \tilde{S}_m]$$

$$=$$

# Regenerative Process

**That is,**

Let $\tilde{W}_t = f(\tilde{Z}_{t+t_1}, \tilde{Z}_{t+t_2}, \ldots, \tilde{Z}_{t+t_n})$.
Let $\hat{\tilde{Z}}_u = \tilde{Z}_{T+u}$ ($\hat{\tilde{Z}}$ is the future process obtained from $\tilde{Z}$ by taking $T = \tilde{S}_m$ as the time origin.)

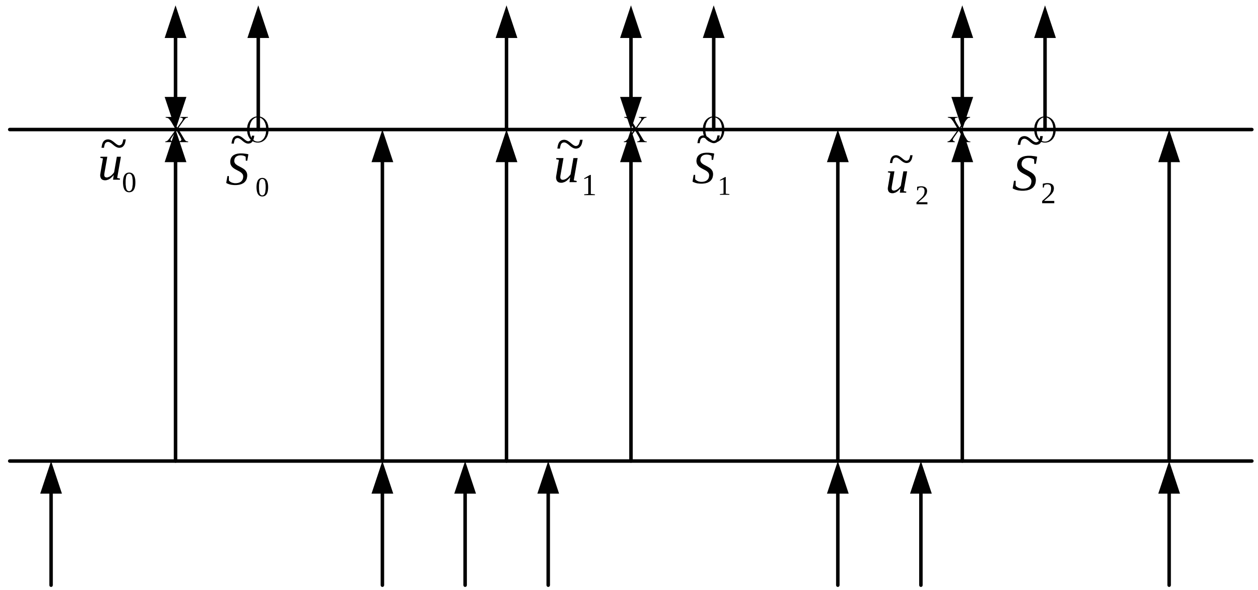
$$\begin{aligned} \therefore \tilde{W}_T &= f(\tilde{Z}_{T+t_1}, \ldots, \tilde{Z}_{T+t_n}) \\ &= f(\tilde{Z}_{t_1}, \ldots, \tilde{Z}_{t_n}) = \hat{\tilde{W}}_0 \end{aligned}$$

Then, the regenerative property says:

1. $E[\hat{\tilde{W}}_0 | Z_u; u < T] = E[\hat{\tilde{W}}_0] \rightarrow$ Future process $\hat{Z}$ is independent of the past history before $T$.
2. $E[\hat{\tilde{W}}_0] = E[\tilde{W}_0] \rightarrow$ Probability law of $\hat{Z}$ is the same as that of $Z$

# Regenerative Process

**Example 1.**




- Let $Z = \{\tilde{Z}_t, t \geq 0\}$, be the queue size at time $t$ for a single sever queueing system, subject to Poisson process of arrivals and General i.i.d. service time distribution ($M/G/1$).

# Regenerative Process

- Time origin = the instant of departure which left behind 0 customers;
- Then, $Z$ is the regenerative process with regeneration time process $S = \{\tilde{S}_n, n \geq 0\}$ (shown as "$\bigcirc$").
- That is, every time a departure occurs leaving behind an empty system, the future of $Z$ after such a time has exactly the same probability law as the process $Z$ starting at time 0.

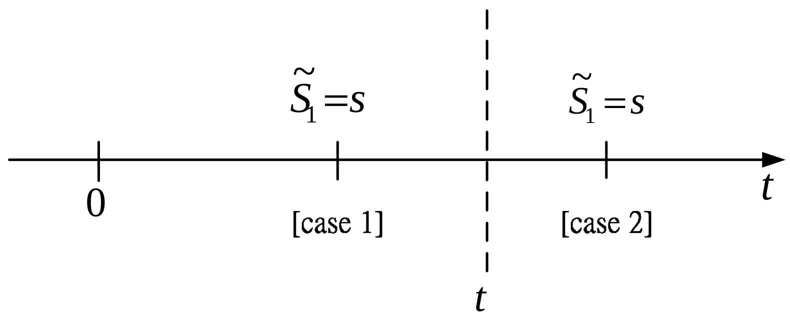
**Example 2.**

- Time origin = the instant of departure leaving behind exactly one customers;
- Then, $Z$ is the regenerative process with regeneration time process $u = \{\tilde{u}_n, n \geq 0\}$ (shown as "$X$").

# Renewal Theory

- The main tool for studying regenerative processes in the absence of future properties
- To study $\tilde{Z}_t = i$ (e.g. number of customers in the system at time $t = i$)
  - $g(t) = P[\tilde{Z}_t = i] =?$ (pdf)
  - $\lim_{t\to\infty} g(t) =?$ (limiting pdf)
- Conditioning the event $\tilde{Z}_t = i$ on the time $\tilde{S}_1$ of the first generation,

$$\because Z \text{ is a regenerative process,}$$
$$\therefore \hat{Z}\ (\triangleq Z_{\tilde{S}_1+t}) \text{ has the same probability law as } Z$$



# Renewal Theory

- Case 1: if $\tilde{S}_1 = s \leq t \Rightarrow$

- Case 2: if $\tilde{S}_1 = s > t \Rightarrow$ ?

- Solving $g(t) \longrightarrow$ solving $h(t)$ ($f_{\tilde{S}}(s)$ is known)
- Solving $\lim_{t\to\infty} g(t) =$? (Key Renewal Theorem !!)

# Renewal Theory

**Example.** Renewal function $m(t) = E[\tilde{n}(t)] =?$

# Renewal Theory

**Question.** How to remove the recursive relationship in the renewal-type equation?

**Solution.** Take Laplace transform and invert it.

# Renewal Theory

**Example 1.**

- $X = \{\tilde{x}_i\}$ i.i.d. inter-arrival time, mean $\bar{X}$,
- Recall: $E[\tilde{S}_{\tilde{N}(t)+1}] = \bar{X}[m(t) + 1]$
- Prove it using Renewal-Type Equation and its solution.

**Answer.**

# Renewal Theory

**Example 2.** Renewal function $m(t)$

$$m(t) = F(t) + \int_0^t m(t-x) f_{\tilde{x}}(x) dx$$

$$\downarrow$$

$$m(t) = F(t) + \int_0^t F(t-x) dm(x)$$

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<**Question**> $\lim_{t\to\infty} g(t) = ?$

# Key Renewal Theorem

**Theorem.** If $F_{\tilde{x}}$ is non-lattice, and if $h(t)$ is directly Riemann integrable (i.e., $h(t) \geq 0$, non-increasing, $\int_0^\infty h(t)dt < \infty$), (integrable with respect to time exists),
then,

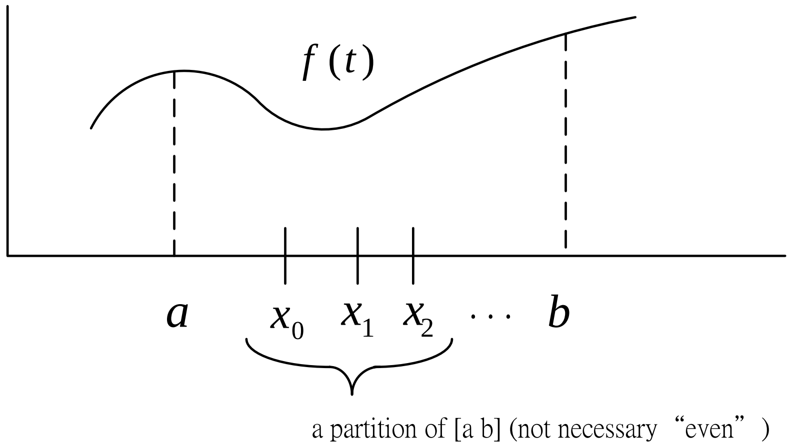
$$\begin{aligned}
\lim_{t\to\infty} g(t) &= \lim_{t\to\infty} \int_0^t h(t-x)dm(x) \\
&= \frac{1}{\bar{X}} \int_0^\infty h(t)dt \\
\text{where} \quad & m(x) = \sum_{n=1}^{\infty} F_n(x) \\
& \bar{X} = \int_0^\infty \bar{F}(x)dx
\end{aligned}$$

**Proof.** (omitted)

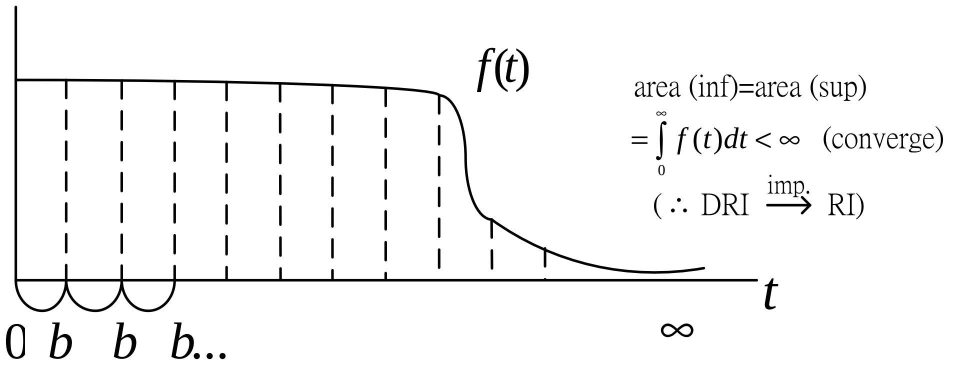
# Key Renewal Theorem

**Note.** Riemann Integral and Directly Riemann Integrable

1. Riemann Integral (RI)




2. Directly Riemann Integrable (DRI)



# Key Renewal Theorem

**Definition.** $f(t)$, defined on $[0, \infty]$, is said to be D.R. Integrable, (defined as $f \in D$), for every $b > 0$, $\overline{m}_n(b)$ and $\underline{m}_n(b)$ be the sup and inf of $f(t)$, i.e.,

$$\overline{m}_n(b) = \sup\{f(t) : nb \leq t < (n+1)b\}$$

$$\underline{m}_n(b) = \inf\{f(t) : nb \leq t < (n+1)b\}$$

if

$$\sum_{n=0}^{\infty} \overline{m}_n(b) \text{ and } \sum_{n=0}^{\infty} \underline{m}_n(b) \text{ are finite, and}$$

$$\lim_{b \to 0} b \cdot \sum_{n=0}^{\infty} \overline{m}_n(b) = \lim_{b \to 0} b \cdot \sum_{n=0}^{\infty} \underline{m}_n(b) = \int_0^{\infty} f(t)dt < \infty$$

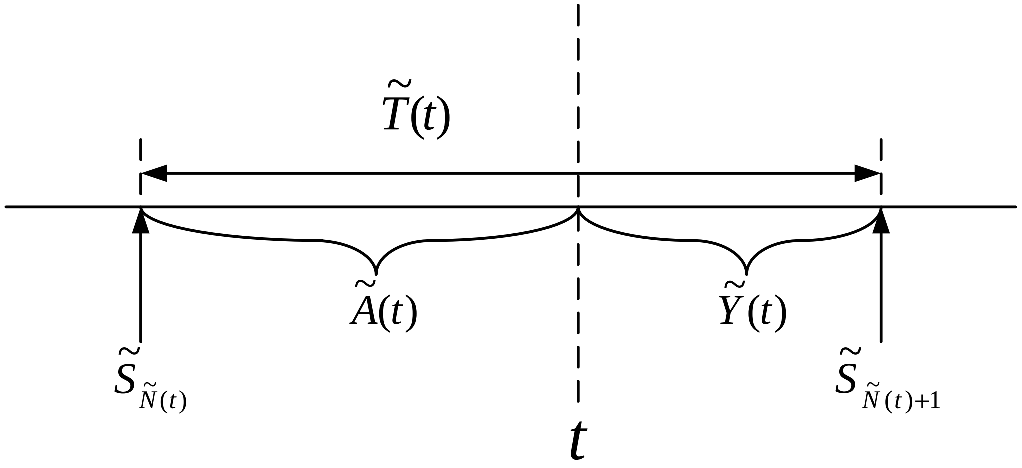
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

# Key Renewal Theorem

- Sufficient conditions for an $f(t)$ to be D.R. Integrable

1. $f(t) \geq 0 \quad \forall t$
2. $f(t)$ non-increasing
3. $\int_0^\infty f(t)dt < \infty$

# Application 1 : Residual Life, Age, and Total Life




- For time $t$,
  - $\tilde{Y}(t) = \tilde{S}_{\tilde{N}(t)+1} - t$ (*Residual Life*, Excess life, *Forward recurrence time*)
  - $\tilde{A}(t) = t - \tilde{S}_{\tilde{N}(t)}$ (*Age*, Current life, *Backward recurrence time*)
  - $\tilde{T}(t) = \tilde{Y}(t) + \tilde{A}(t) = \tilde{x}_{\tilde{N}(t)+1}$ (*life*, spread, *recurrence time*)

# Application 1 : Residual Life, Age, and Total Life

**To find:** $(\tilde{Y}(t))$

- $F_{\tilde{Y}(t)}(x) = ?$ $(\bar{F}_{\tilde{Y}(t)}(x) = ?)$ (Renewal-Type Equation & solution)
- $\lim_{t\to\infty} F_{\tilde{Y}(t)}(x) = ?$ (Key Renewal Theorem)
- $\lim_{t\to\infty} E[\tilde{Y}(t)] = ?$ $(F_{\tilde{Y}(t)})$

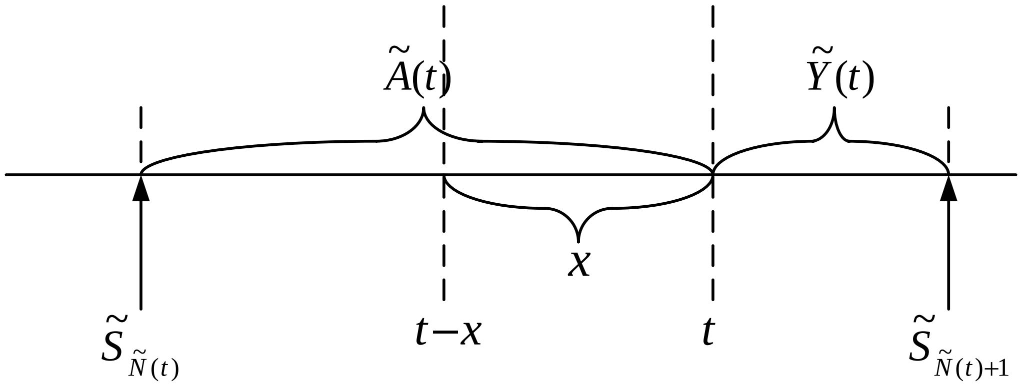
# Application 1 : Residual Life, Age, and Total Life

$$\begin{aligned}\lim_{t\to\infty} E[\tilde{Y}(t)] &= \lim_{t\to\infty}\int_0^\infty \bar{F}_{\tilde{Y}(t)}(x)dx \\ &=\end{aligned}$$

# Application 1 : Residual Life, Age, and Total Life

**To find:** $(\tilde{A}(t))$

- $F_{\tilde{A}(t)}(x) = ?$ $(\bar{F}_{\tilde{A}(t)}(x) = ?)$




Notice that :

- $\tilde{A}(t) > x \Leftrightarrow$
- $P(\tilde{A}(t) > x) = 0$, where

# Application 1 : Residual Life, Age, and Total Life

**<Homework>**

1. Find $\lim_{t \to \infty} F_{\tilde{A}(t)}(x) = ?$
2. Find $\lim_{t \to \infty} E[\tilde{A}(t)] = ?$

# Application 1 : Residual Life, Age, and Total Life

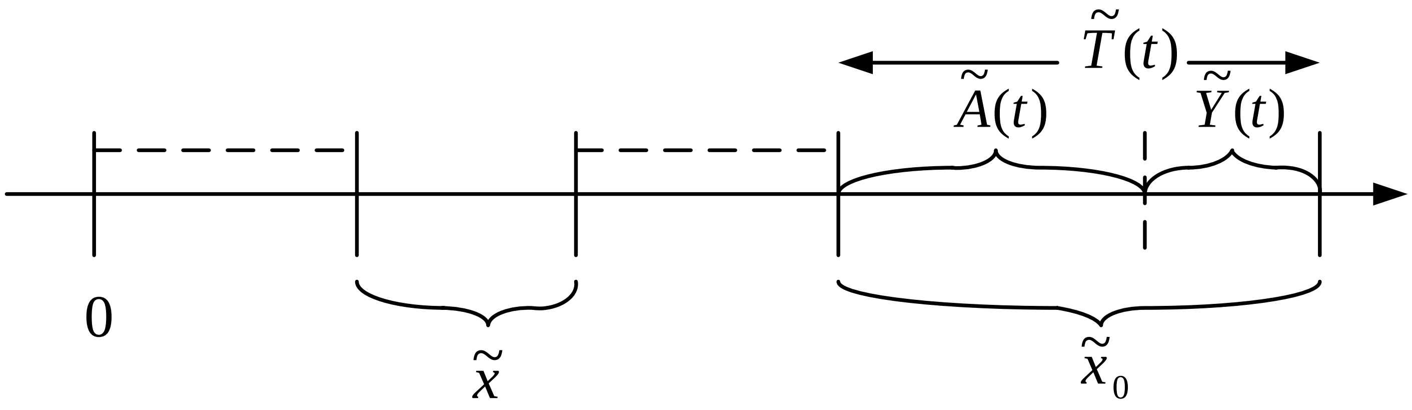
**To find:** $\tilde{T}(t)$

- $F_{\tilde{T}(t)}(x) = ?$
- $\lim_{t \to \infty} F_{\tilde{T}(t)}(x) = ?$

# Application 1 : Residual Life, Age, and Total Life

$<$**Homework.**$>$ Find $\lim_{t \to \infty} E[\tilde{T}(t)] = ?$

# The Inspection Paradox




$$\tilde{T}(t) = \tilde{S}_{\tilde{N}(t)+1} - \tilde{S}_{\tilde{N}(t)} = \tilde{X}_{\tilde{N}(t)+1} \triangleq \tilde{x}_0$$

From above, we get: $F_{\tilde{x}_0}(x) = \dfrac{1}{E[\tilde{x}]} \cdot \int_0^x y \cdot dF_{\tilde{x}}(y)$

From definition, we get: $F_{\tilde{x}}(x) = \int_0^x dF_{\tilde{x}}(y)$

Why $F_{\tilde{x}_0}(x) \neq F_{\tilde{x}}(x)$ ?

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

## The Inspection Paradox

- That is, the length of the renewal interval containing $t$ is stochastically greater than the length of an ordinary renewal interval
  - If you drop a point to a segmented time line, the segment that the point falls into should be larger than other segments
  - "Inspection paradox" [Ref. Ross, P.118-Remark]

# Application 2 : Alternating Renewal Process

What is the distribution of $\tilde{S}_{\tilde{N}(t)}$, i.e., the time of the last renewal prior to (or at) time $t$ (will be used later)?

**Lemma.**

$$P[\tilde{S}_{\tilde{N}(t)} \leq s] = \bar{F}_{\tilde{x}_1}(t) + \int_0^s \bar{F}_{\tilde{x}_1}(t-y)dm(y), \ s \leq t$$

**Proof.**

# Application 2 : Alternating Renewal Process

Note: From the previous lemma, we get:

$$
\begin{aligned}
P[\tilde{S}_{\tilde{N}(t)} = 0] &= \bar{F}_{\tilde{x}_1}(t) \\
dF_{\tilde{S}_{\tilde{N}(t)}}(y) &= \bar{F}_{\tilde{x}_1}(t-y)dm(y)
\end{aligned}
$$

$\downarrow$ reasoning

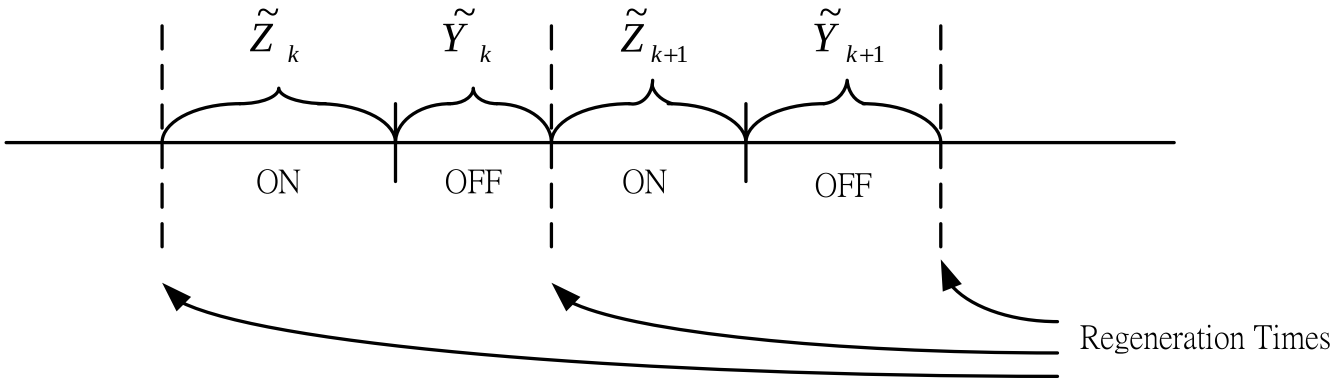
$$
\begin{aligned}
dF_{\tilde{S}_{\tilde{N}(t)}}(y) &= f_{\tilde{S}_{\tilde{N}(t)}}(y)dy \\
&=
\end{aligned}
$$

# Application 2 : Alternating Renewal Process

$\downarrow$ To prove:

Alternating Renewal Theory (Conditioning on $\tilde{S}_{\tilde{N}(t)}$)

# Alternating Renewal Processes




$\{(\tilde{Z}_k, \tilde{Y}_k), k \geq 1\}$ are i.i.d.

$$\Rightarrow \text{Alternating Renewal Processes} \begin{cases} \tilde{Z}_i \sim F_{\tilde{Z}}(t) \\ \tilde{Y}_i \sim F_{\tilde{Y}}(t) \\ \tilde{Z}_i + \tilde{Y}_i \sim F_{\tilde{X}}(t) \end{cases}$$

**Theorem.** If $E[\tilde{Z}_n + \tilde{Y}_n] < \infty$, and $F_{\tilde{Z}_n+\tilde{Y}_n}$ is non-arithmetic, then

$$\lim_{t\to\infty} P[\text{system is ``ON'' at time } t] \triangleq \lim_{t\to\infty} P(t) = \frac{E[\tilde{Z}_n]}{E[\tilde{Z}_n] + E[\tilde{Y}_n]}$$

# Alternating Renewal Processes

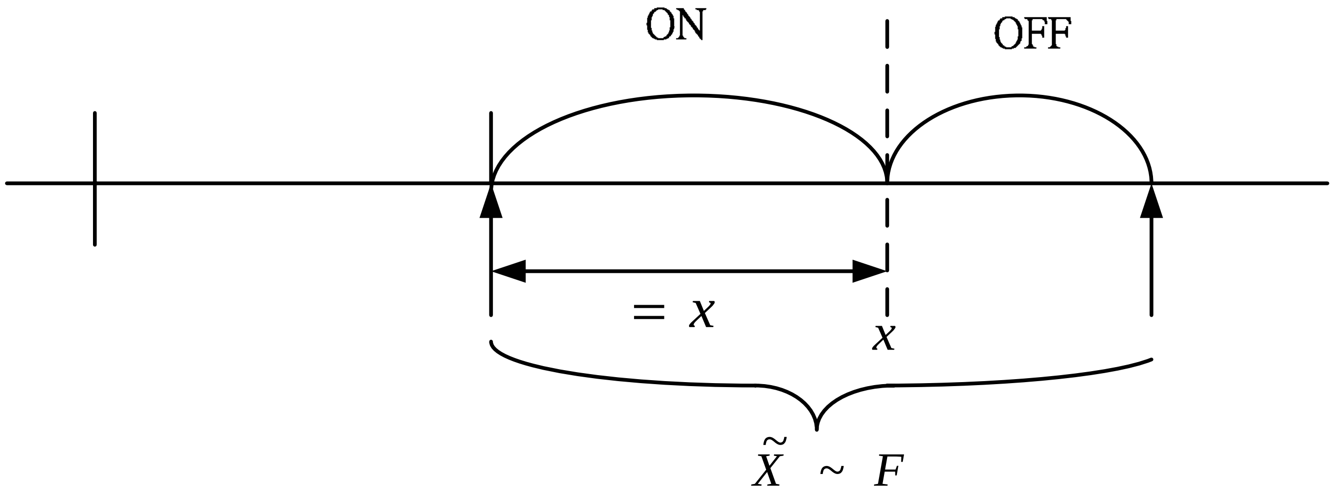
**Proof.**

# Applications of the Alternating Renewal Theory

Computation of the distributions of $\tilde{A}(t), \tilde{Y}(t)$, and $\tilde{T}(t)$, i.e.,

$\lim_{t\to\infty} P[\tilde{A}(t) \le x] =?$ $(\lim_{t\to\infty} P[\tilde{Y}(t) \le x] =?)$ $(\lim_{t\to\infty} P[\tilde{T}(t) \le x] =?)$

1. - Let an on-off cycle correspond to a renewal interval.
   - The system is "on" at time $t$ if the age at $t$ is less or equal to $x$, i.e., "on" the first $x$ units of a renewal interval.



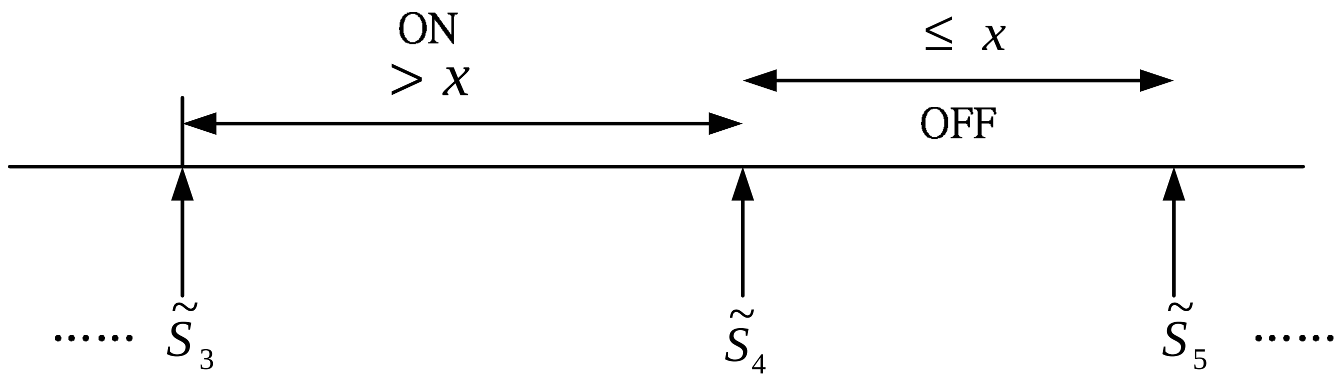
# Applications of the Alternating Renewal Theory

2.

$$
\begin{aligned}
\lim_{t\to\infty} P[\tilde{Y}(t) \le x] &= \lim_{t\to\infty} P[\text{“OFF” at } t] \\
&=
\end{aligned}
$$

# Applications of the Alternating Renewal Theory

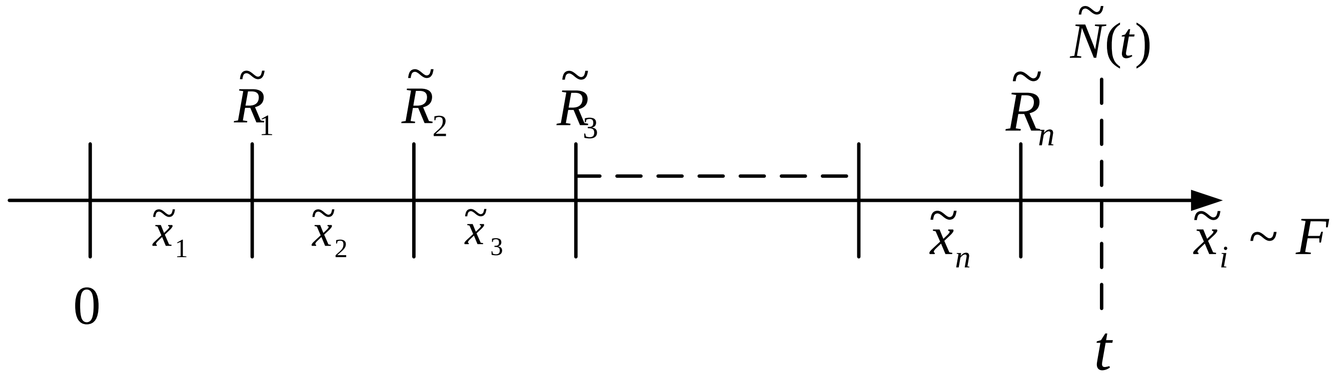
3. Consider $\begin{cases} \text{cycle time } \tilde{x} > x & \rightarrow & \text{“ON”} \\ \text{cycle time } \tilde{x} \leq x & \rightarrow & \text{“OFF”} \end{cases}$



## Application 3 : Compute $E[\tilde{Y}(t)]$ by conditioning $\tilde{S}_{\tilde{N}(t)}$

$$\begin{aligned} E[\tilde{Y}(t)] &= E[\tilde{Y}(t)|\tilde{S}_{\tilde{N}(t)} = 0] \cdot \bar{F}(t) + \int_0^t E[\tilde{Y}(t)|\tilde{S}_{\tilde{N}(t)} = y]\bar{F}(t-y)dm(y) \\ &= \end{aligned}$$

# Renewal Reward Process and Applications




- $\tilde{R}_n \triangleq$ the reward earned at the time of the $n_{th}$ renewal;
- $\tilde{R}_n \geq 0$, for all $n$;
- $\{\tilde{R}_n, n \geq 1\}$ are i.i.d., with mean $E[\tilde{R}]$;
- $\tilde{R}_n$ may depend on $\tilde{x}_n$;
- $\therefore \{(\tilde{R}_n, \tilde{x}_n), n \geq 1\}$ i.i.d. random variables;
- Let $\tilde{R}(t) = \sum_{n=1}^{\tilde{N}(t)} \tilde{R}_n \triangleq$ the total reward earned by $t$

# Renewal Reward Process and Applications

**Theorem.** If $E[\tilde{R}] < \infty,\ E[\tilde{x}] < \infty$, then

1.

$$\frac{\tilde{R}(t)}{t} \rightarrow \frac{E[\tilde{R}]}{E[\tilde{x}]} \text{ w.p.1 as } t \rightarrow \infty$$

i.e., long-run average reward =

2.

$$\frac{E[\tilde{R}(t)]}{t} \rightarrow \frac{E[\tilde{R}]}{E[\tilde{x}]} \text{ as } t \rightarrow \infty$$

i.e., expected long-run average reward =

# Renewal Reward Process and Applications

Note: The *Renewal Reward Theorem* says that:

$$\frac{\tilde{R}(t)}{t} \to \frac{E[\tilde{R}]}{E[\tilde{x}]} \text{ w.p.1 as } t \to \infty, \text{ i.e., } \lim_{t\to\infty} \underbrace{\frac{\sum_{n=1}^{\tilde{N}(t)} \tilde{R}_n}{t}}_{\text{Time Average}} = \frac{E[\tilde{R}]}{E[\tilde{x}]}$$

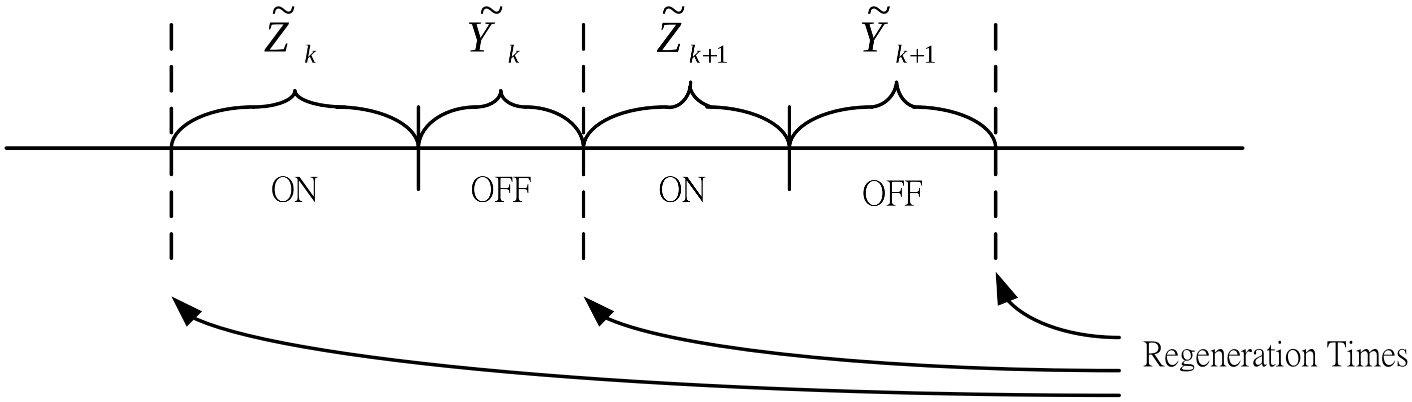
$\therefore$ The long-run average reward

$=$

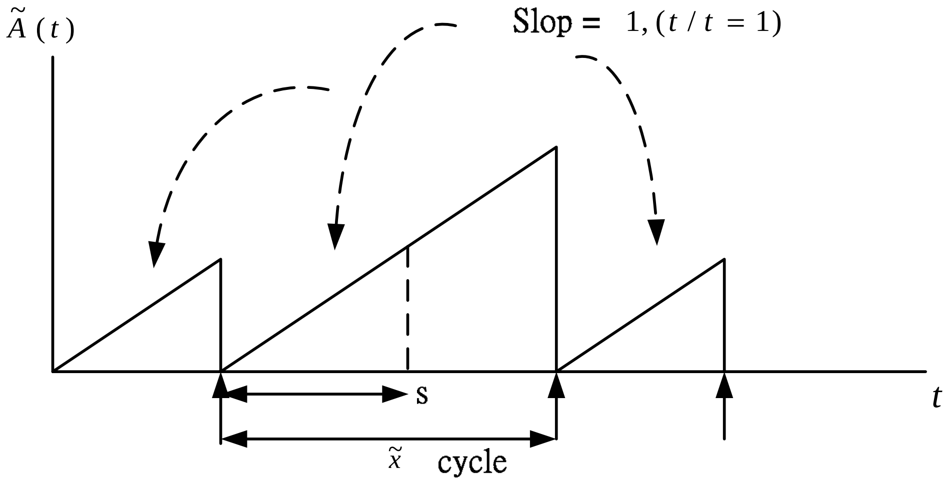
$=$

$=$

# Application # 1 : (Alternating Renewal Processes)

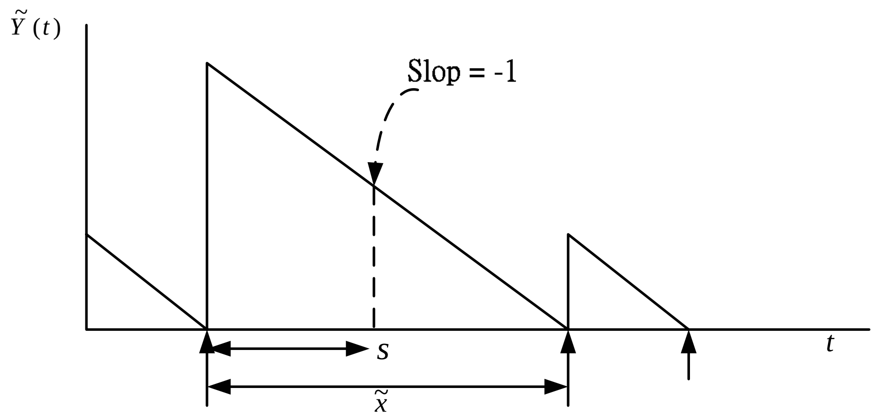


# Application # 2 : (Time Avg. of Age and Residual life)




$$\text{To find } \lim_{t\to\infty} \frac{\int_0^t \tilde{A}(s)ds}{t} = ?$$

# Application # 2 : (Time Avg. of Age and Residual life)




To find $\lim_{t\to\infty} \dfrac{\int_0^t \tilde{Y}(s)ds}{t}$ =? Note that $\tilde{Y}(s) = \tilde{x} - s$.

# Application # 3 : The Little's Formula – Part I

- A $G/G/1$ queueing server:
  - Let $X_1, X_2, \ldots$ denote the interarrival times between customers; and let $Y_1, Y_2, \ldots$ denote the service times of successive customers. We shall assume that

$$E[Y_i] < E[X_i] < \infty$$

- Suppose that the first customer arrives at time 0 and let $n(t)$ denote the number of customers in the system at time $t$. Define

$$L = \lim_{t \to \infty} \int_0^t n(s) ds / t$$

- Imagine that a reward is being earned at time $s$ at rate $n(s)$. If we let a cycle correspond to the start of a busy period, then the process restarts itself each cycle.

# Application # 3 : The Little's Formula – Part I

- As $L$ represents the long-run average reward, it follows from the Renewal Reward Theorem that

$$\begin{aligned} L &= \\ &= \end{aligned}$$

# Application # 3 : The Little's Formula – Part II

- Let $W_i$ denote the amount of time the $i$th customer spends in the system and define

$$W = \lim_{n\to\infty} \frac{W_1 + \cdots + W_n}{n}$$

- Let $N$ denote the number of customers served in a cycle, then $W$ is the average reward per unit time of a renewal process in which the cycle time is $N$ and the cycle reward is $W_1 + \cdots + W_N$, and, hence,

$$\begin{aligned} W &= \\ &= \end{aligned}$$

## Application # 3 : The Little's Formula – Part III

**Theorem.** Let $\lambda = 1/E[X_i]$ denote the arrival rate. Then

$$L = \lambda W$$

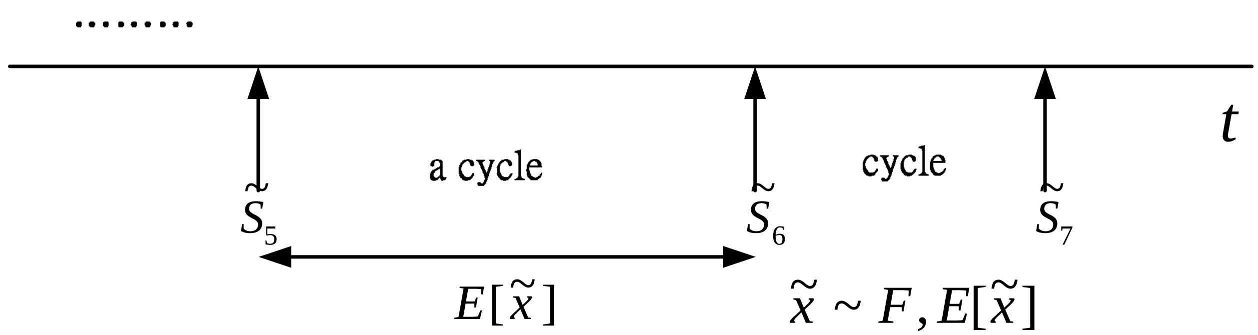
**Proof.**

# Application # 3 : The Little's Formula – Part III

**Remarks**

- The Little's Formula states that
- By replacing "the system" by "the queue" the same proof shows that
- By replacing "the system" by "service" we have that

# Regenerative Processes




Stochastic process $Z = \{\tilde{Z}(t), t \geq 0\}$ with state space $S = \{0, 1, 2, \ldots\}$ is called a *regenerative process* if the regenerative property holds.

**Theorem.** If $E[\tilde{x}] < \infty$

$$\begin{aligned}
\lim_{t \to \infty} P[\tilde{Z}(t) = j] &= \frac{E[\text{amount of time in state } j \text{ in a cycle}]}{E[\text{cycle length}]} \\
&= \frac{\int_0^\infty P[\tilde{Z}(t) = j, \tilde{x}_1 > t] dt}{E[\tilde{x}]}
\end{aligned}$$

# Regenerative Processes

**Proof.**

# Regenerative Processes

**Theorem.** For a regenerative process with $E[\tilde{x}_1] < \infty$, with probability 1,

$$\lim_{t \to \infty} \frac{[\text{time in } j \text{ during } (0,t)]}{t} = \frac{E[\text{time in state } j \text{ during a cycle}]}{E[\text{time of a cycle}]}$$

**Proof.**

**Homework.** to be announced on the web

# Delayed Renewal Processes

- We often consider a counting process for which the first interarrival time has a different distribution from the remaining ones.
- For instance, we might start observing a renewal process at some time $t > 0$. If a renewal does not occur at $t$, then the distribution of the time we must wait until the first observed renewal will not be the same as the remaining interarrival distributions.
- Formally, let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent nonnegative random variables with $X_1$ having distribution $G$, and $X_n$ having distribution $F$, $n > 1$. Let $S_0 = 0$, $S_n = \sum_1^n X_i, n \geq 1$, and define
$$N_D(t) = \sup\{n : S_n \leq t\}.$$
- **Definition.** The stochastic process $\{N_D(t), t \geq 0\}$ is called a *general* or a *delayed* renewal process.

# Delayed Renewal Processes

- When $G = F$, we have, of course, an ordinary renewal process. As in the ordinary case, we have

$$
\begin{aligned}
P\{N_D(t) = n\} &= \\
&=
\end{aligned}
$$

- Let $m_D(t) = E[N_D(t)]$. Then it is easy to show that

$$m_D(t) =$$

and by taking transforms, we obtain

$$\tilde{m}_D(s) =$$

# Delayed Renewal Processes

By using the corresponding result for the ordinary renewal process, it is easy to prove similar limit theorems for the delayed process. Let $\mu = \int_0^\infty x dF(x)$.

**Proposition.**

1. With probability 1,

$$\frac{N_D(t)}{t} \to \frac{1}{\mu} \qquad \text{as } t \to \infty$$

2.

$$\frac{m_D(t)}{t} \to \frac{1}{\mu} \qquad \text{as } t \to \infty$$

# Delayed Renewal Processes

3. If $F$ is not lattice, then

$$m_D(t+a) - m_D(t) \to \frac{a}{\mu} \qquad \text{as } t \to \infty$$

4. If $F$ and $G$ are lattice with period $d$, then

$$E[\text{number of renewals at } nd] \to \frac{d}{\mu} \qquad \text{as } n \to \infty$$

5. If $F$ is not lattice, $\mu < \infty$, and $h$ directly Riemann integrable, then

$$\int_0^\infty h(t-x)dm_D(x) \to \int_0^\infty h(t)dt/\mu$$

# Delayed Renewal Processes

- In the same way we proved the result in the case of an ordinary renewal process, it follows that the distribution of the time of the last renewal before (or at) $t$ is given by

$$P\{S_{N(t)} \leq s\} =$$

- When $\mu < \infty$, the distribution function

$$F_e(x) =$$

  is called the *equilibrium distribution* of $F$. Its Laplace transform is given by

$$\begin{aligned} \tilde{F}_e(s) &= \int_0^\infty e^{-sx} dF_e(x) \\ &= \end{aligned}$$

# Delayed Renewal Processes

- The delayed renewal process with $G = F_e$ is called the *equilibrium renewal process* and is extremely important.
- For suppose that we start observing a renewal process at time $t$. Then the process we observe is a delayed renewal process whose initial distribution is the distribution of $Y(t)$ (i.e., residual life). Thus, for $t$ large, it follows that the observed process is the equilibrium renewal process.

# Delayed Renewal Processes

Let $Y_D(t)$ denote the residual life at $t$ for a delayed renewal process.

**Theorem.** For the equilibrium renewal process:

1. $m_D(t) = t/\mu$
2. $P\{Y_D(t) \leq x\} = F_e(x)$ for all $t \geq 0$
3. $\{N_D(t), t \geq 0\}$ has stationary increments

**Proof.**

# Delayed Renewal Processes