Chapter 2. Poisson Processes

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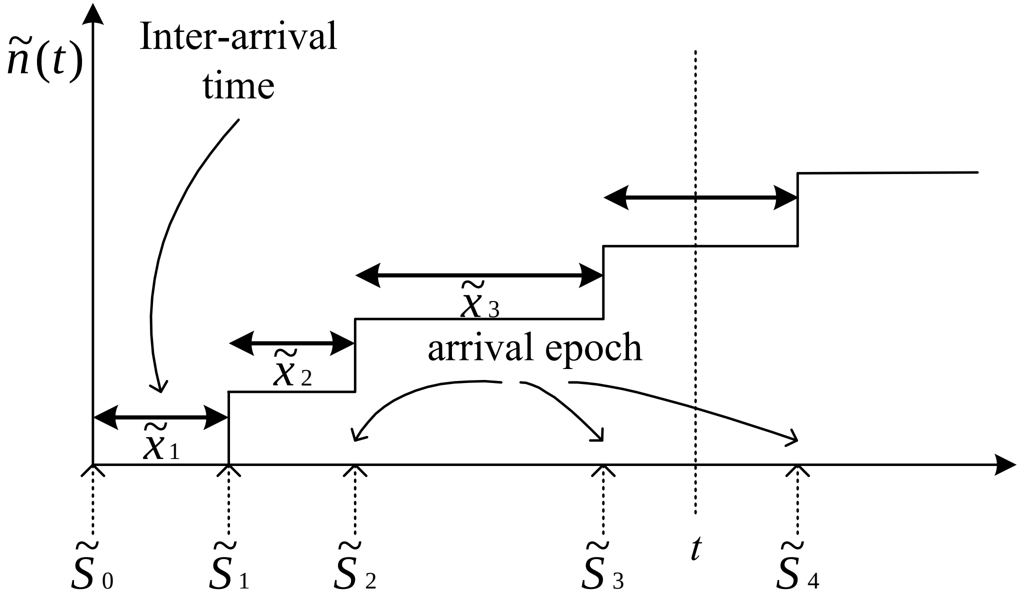
Outline

- Introduction to Poisson Processes
  - Definition of arrival process
  - Definition of renewal process
  - Definition of Poisson process
- Properties of Poisson processes
  - Inter-arrival time distribution
  - Waiting time distribution
  - Superposition and decomposition
- Non-homogeneous Poisson processes (relaxing *stationary*)
- Compound Poisson processes (relaxing *single arrival*)
- Modulated Poisson processes (relaxing *independent*)
- Poisson Arrival See Average (PASTA)

Introduction



(i) n_{th} arrival epoch $\tilde{S}_n$ is

$$\tilde{S}_n = \tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_n = \sum_{i=1}^{n} \tilde{x}_i$$

$$\tilde{S}_0 = 0$$

(ii) Number of arrivals at time t is: $\tilde{n}(t)$. Notice that:

$$\{\tilde{n}(t) \geq n\} \overset{iff}{\Leftrightarrow} \{\tilde{S}_n \leq t\}, \{\tilde{n}(t) = n\} \overset{iff}{\Leftrightarrow} \{\tilde{S}_n \leq t \text{ and } \tilde{S}_{n+1} > t\}$$

Introduction

Arrival Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\};\ \tilde{x}_i$'s can be any

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}; \tilde{S}_i$'s can be any

$N = \{\tilde{n}(t),\ t \geq 0\}; \longrightarrow$ called arrival process

Renewal Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\};\ \tilde{x}_i$'s are i.i.d.

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}; \tilde{S}_i$'s are general distributed

$N = \{\tilde{n}(t),\ t \geq 0\}; \longrightarrow$ called renewal process

Poisson Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\};\ \tilde{x}_i$'s are iid exponential distributed

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}; \tilde{S}_i$'s are Erlang distributed

$N = \{\tilde{n}(t),\ t \geq 0\}; \longrightarrow$ called Poisson process

Counting Processes

- A stochastic process $N = \{\tilde{n}(t),\, t \geq 0\}$ is said to be a *counting process* if $\tilde{n}(t)$ represents the total number of "events" that have occurred up to time t.

- From the definition we see that for a counting process $\tilde{n}(t)$ must satisfy:

  1. $\tilde{n}(t) \geq 0$.
  2. $\tilde{n}(t)$ is integer valued.
  3. If $s < t$, then $\tilde{n}(s) \leq \tilde{n}(t)$.
  4. For $s < t$, $\tilde{n}(t) - \tilde{n}(s)$ equals the number of events that have occurred in the interval $(s, t]$.

Definition 1: Poisson Processes

The counting process $N = \{\tilde{n}(t),\ t \geq 0\}$ is a *Poisson process* with rate λ ($\lambda > 0$), if:

1. $\tilde{n}(0) = 0$
2. Independent increments $\boxed{\text{relaxed} \Rightarrow \text{Modulated Poisson Process}}$

$$P[\tilde{n}(t) - \tilde{n}(s) = k_1 | \tilde{n}(r) = k_2,\ r \leq s < t] = P[\tilde{n}(t) - \tilde{n}(s) = k_1]$$

3. Stationary increments $\boxed{\text{relaxed} \Rightarrow \text{Non-homogeneous Poisson Process}}$

$$P[\tilde{n}(t+s) - \tilde{n}(t) = k] = P[\tilde{n}(l+s) - \tilde{n}(l) = k]$$

4. Single arrival $\boxed{\text{relaxed} \Rightarrow \text{Compound Poisson Process}}$

$$\begin{aligned} P[\tilde{n}(h) = 1] &= \lambda h + o(h) \\ P[\tilde{n}(h) \geq 2] &= o(h) \end{aligned}$$

Definition 2: Poisson Processes

The counting process $N = \{\tilde{n}(t),\ t \geq 0\}$ is a *Poisson process* with rate λ ($\lambda > 0$), if:

1. $\tilde{n}(0) = 0$
2. Independent increments
3. The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$

$$P[\tilde{n}(t+s) - \tilde{n}(s) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Theorem: Definitions 1 and 2 are equivalent.

Proof. We show that Definition 1 implies Definition 2, and leave it to the reader to prove the reverse. To start, fix $u \geq 0$ and let

$$g(t) = E[e^{-u\tilde{n}(t)}]$$

We derive a differential equation for $g(t)$ as follows:

$$\begin{aligned} g(t+h) &= E[e^{-u\tilde{n}(t+h)}] \\ &= E\left\{e^{-u\tilde{n}(t)}e^{-u[\tilde{n}(t+h)-\tilde{n}(t)]}\right\} \\ &= E\left[e^{-u\tilde{n}(t)}\right]E\left\{e^{-u[\tilde{n}(t+h)-\tilde{n}(t)]}\right\} \quad \text{by independent increments} \\ &= g(t)E\left[e^{-u\tilde{n}(h)}\right] \quad \text{by stationary increments} \end{aligned} \tag{1}$$

Theorem: Definitions 1 and 2 are equivalent.

Conditioning on whether $\tilde{n}(t) = 0$ or $\tilde{n}(t) = 1$ or $\tilde{n}(t) \geq 2$ yields

$$\begin{aligned} E\left[e^{-u\tilde{n}(h)}\right] &= 1 - \lambda h + o(h) + e^{-u}(\lambda h + o(h)) + o(h) \\ &= 1 - \lambda h + e^{-u}\lambda h + o(h) \qquad (2) \end{aligned}$$

From (1) and (2), we obtain that

$$g(t+h) = g(t)(1 - \lambda h + e^{-u}\lambda h) + o(h)$$

implying that

$$\frac{g(t+h) - g(t)}{h} = g(t)\lambda(e^{-u} - 1) + \frac{o(h)}{h}$$

Theorem: Definitions 1 and 2 are equivalent.

Letting $h \to 0$ gives

$$g'(t) = g(t)\lambda(e^{-u} - 1)$$

or, equivalently,

$$\frac{g'(t)}{g(t)} = \lambda(e^{-u} - 1)$$

Integrating, and using $g(0) = 1$, shows that

$$\log(g(t)) = \lambda t(e^{-u} - 1)$$

or

$$g(t) = e^{\lambda t(e^{-u}-1)} \quad \longrightarrow \quad \text{the Laplace transform of a Poisson r. v.}$$

Since $g(t)$ is also the Laplace transform of $\tilde{n}(t)$, $\tilde{n}(t)$ is a Poisson r. v.

The Inter-Arrival Time Distribution

Theorem. Poisson Processes have exponential inter-arrival time distribution, i.e., $\{\tilde{x}_n, n = 1, 2, \ldots\}$ are i.i.d and exponentially distributed with parameter λ (i.e., mean inter-arrival time $= 1/\lambda$).

Proof.

$$\begin{aligned}
\tilde{x}_1 \quad &: \quad P(\tilde{x}_1 > t) = P(\tilde{n}(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t} \\
& \quad \therefore \tilde{x}_1 \sim e(t; \lambda) \\
\tilde{x}_2 \quad &: \quad P(\tilde{x}_2 > t | \tilde{x}_1 = s) \\
& \quad = P\{0 \text{ arrivals in } (s, s+t] | \tilde{x}_1 = s\} \\
& \quad = P\{0 \text{ arrivals in } (s, s+t]\} (\text{by independent increment}) \\
& \quad = P\{0 \text{ arrivals in } (0, t]\} (\text{by stationary increment}) \\
& \quad = e^{-\lambda t} \quad \therefore \tilde{x}_2 \text{ is independent of } \tilde{x}_1 \text{ and } \tilde{x}_2 \sim exp(t; \lambda).
\end{aligned}$$

$\Rightarrow$ The procedure repeats for the rest of $\tilde{x}_i$'s.

The Arrival Time Distribution of the nth Event

Theorem. The arrival time of the n_{th} event, $\tilde{S}_n$ (also called the waiting time until the n_{th} event), is *Erlang* distributed with parameter (n, λ).

Proof. Method 1 :

$$\because \quad P[\tilde{S}_n \le t] = P[\tilde{n}(t) \ge n] = \sum_{k=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$\therefore \quad f_{\tilde{S}_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \quad \text{(exercise)}$$

Method 2 :

$$\begin{aligned} f_{\tilde{S}_n}(t)dt &= dF_{\tilde{S}_n}(t) = P[t < \tilde{S}_n < t + dt] \\ &= P\{n-1 \text{ arrivals in } (0,t] \text{ and } 1 \text{ arrival in } (t, t+dt)\} + o(dt) \\ &= P[\tilde{n}(t) = n-1 \text{ and } 1 \text{ arrival in } (t, t+dt)] + o(dt) \\ &= P[\tilde{n}(t) = n-1]P[1 \text{ arrival in } (t, t+dt)] + o(dt)(why?) \end{aligned}$$

The Arrival Time Distribution of the nth Event

$$
\begin{aligned}
&= \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\lambda dt + o(dt) \\
\therefore\ & \lim_{dt \to 0} \frac{f_{\tilde{S}_n}(t)dt}{dt} = f_{\tilde{S}_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}
\end{aligned}
$$

Conditional Distribution of the Arrival Times

Theorem. Given that $\tilde{n}(t) = n$, the n arrival times $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_n$ have the same distribution as the order statistics corresponding to n i.i.d. uniformly distributed random variables from $(0, t)$.

. . . .

Order Statistics. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be n i.i.d. continuous random variables having common pdf f. Define $\tilde{x}_{(k)}$ as the k_{th} smallest value among all $\tilde{x}_i$'s, i.e., $\tilde{x}_{(1)} \le \tilde{x}_{(2)} \le \tilde{x}_{(3)} \le \ldots \le \tilde{x}_{(n)}$, then $\tilde{x}_{(1)}, \ldots, \tilde{x}_{(n)}$ are known as the "order statistics" corresponding to random variables $\tilde{x}_1, \ldots, \tilde{x}_n$. We have that the joint pdf of $\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)}$ is

$$f_{\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)}}(x_1, x_2, \ldots, x_n) = n! f(x_1) f(x_2) \ldots f(x_n),$$

where $x_1 < x_2 < \ldots < x_n$ (check the textbook [Ross]).

. . . .

Conditional Distribution of the Arrival Times

Proof. Let $0 < t_1 < t_2 < \ldots < t_{n+1} = t$ and let h_i be small enough so that $t_i + h_i < t_{i+1}$, $i = 1, \ldots, n$.

$$\because \quad P[t_i < \tilde{S}_i < t_i + h_i,\ i = 1, \ldots, n | \tilde{n}(t) = n]$$

$$= \frac{P\left(\begin{array}{c}\text{exactly one arrival in each } [t_i, t_i + h_i] \\ i = 1, 2, \ldots, n,\ \text{and no arrival elsewhere in } [0, t]\end{array}\right)}{P[\tilde{n}(t) = n]}$$

$$= \frac{(e^{-\lambda h_1}\lambda h_1)(e^{-\lambda h_2}\lambda h_2)\ldots(e^{-\lambda h_n}\lambda h_n)(e^{-\lambda(t-h_1-h_2\ldots-h_n)})}{e^{-\lambda t}(\lambda t)^n/n!}$$

$$= \frac{n!(h_1 h_2 h_3 \ldots h_n)}{t^n}$$

$$\therefore \quad \frac{P[t_i < \tilde{S}_i < t_i + h_i,\ i = 1, \ldots, n | \tilde{n}(t) = n]}{h_1 h_2 \ldots h_n} = \frac{n!}{t^n}$$

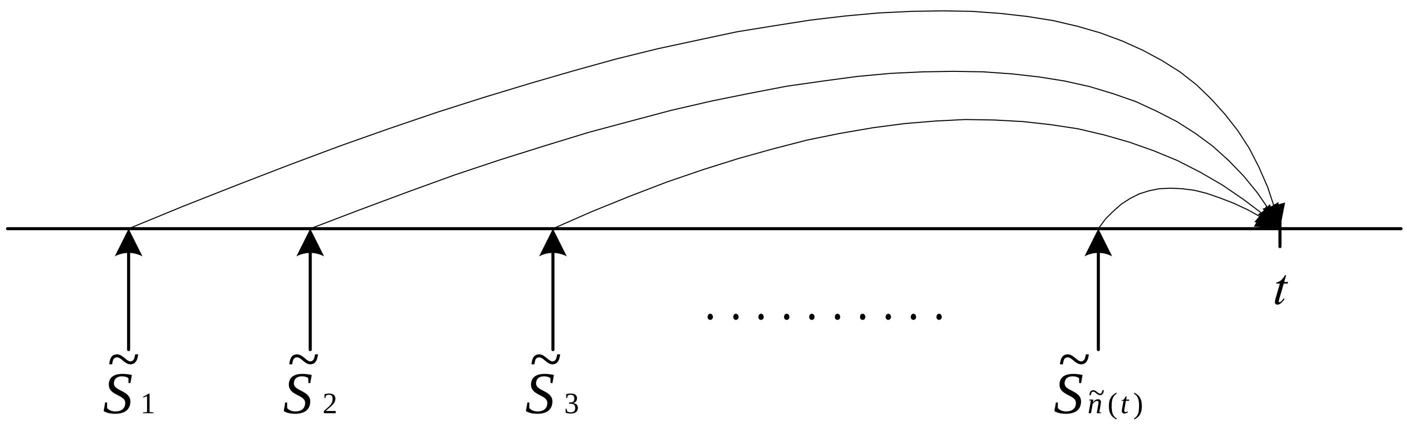
Conditional Distribution of the Arrival Times

Taking $\lim_{h_i \to 0, i=1,\ldots,n} (\quad)$, then

$$f_{\tilde{S}_1,\tilde{S}_2,\ldots,\tilde{S}_n|\tilde{n}(t)}(t_1,t_2,\ldots,t_n|n) = \frac{n!}{t^n},\ 0 < t_1 < t_2 < \ldots < t_n.$$

Conditional Distribution of the Arrival Times

Example (see Ref [Ross], Ex. 2.3(A) p.68). Suppose that travellers arrive at a train depot in accordance with a Poisson process with rate λ. If the train departs at time t, what is the expected sum of the waiting times of travellers arriving in $(0, t)$? That is, $E[\sum_{i=1}^{\tilde{n}(t)}(t - \tilde{S}_i)] = ?$



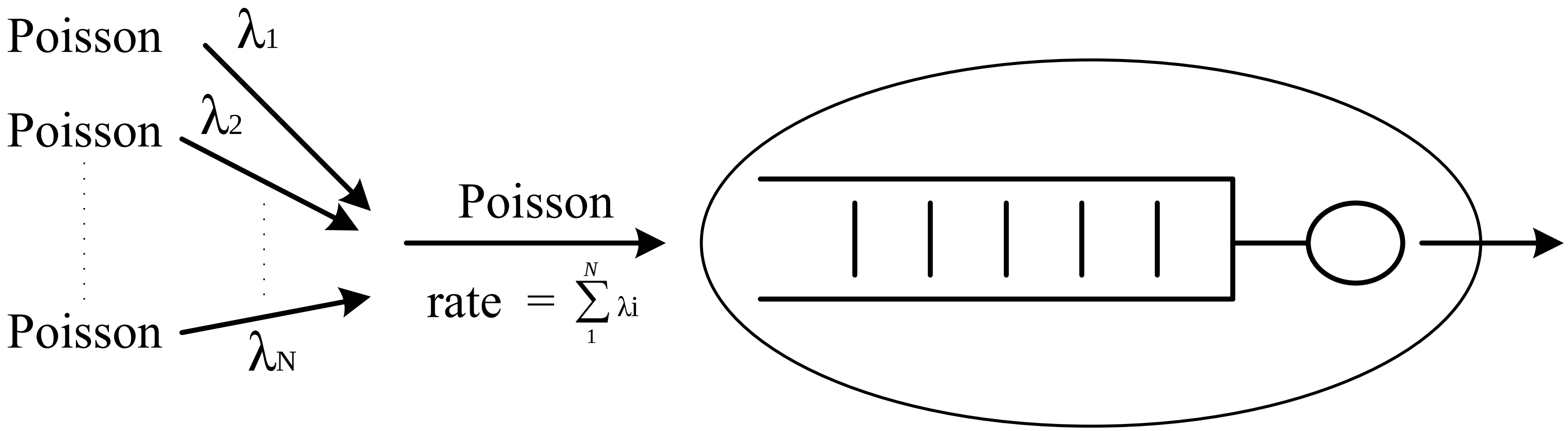
Conditional Distribution of the Arrival Times

Answer.

Superposition of Independent Poisson Processes

Theorem. Superposition of independent Poisson Processes

$(\lambda_i, i = 1, \ldots, N)$, is also a Poisson process with rate $\sum_1^N \lambda_i$.



<u><**Homework**></u> Prove the theorem (note that a Poisson process must satisfy Definitions 1 or 2).

Decomposition of a Poisson Process

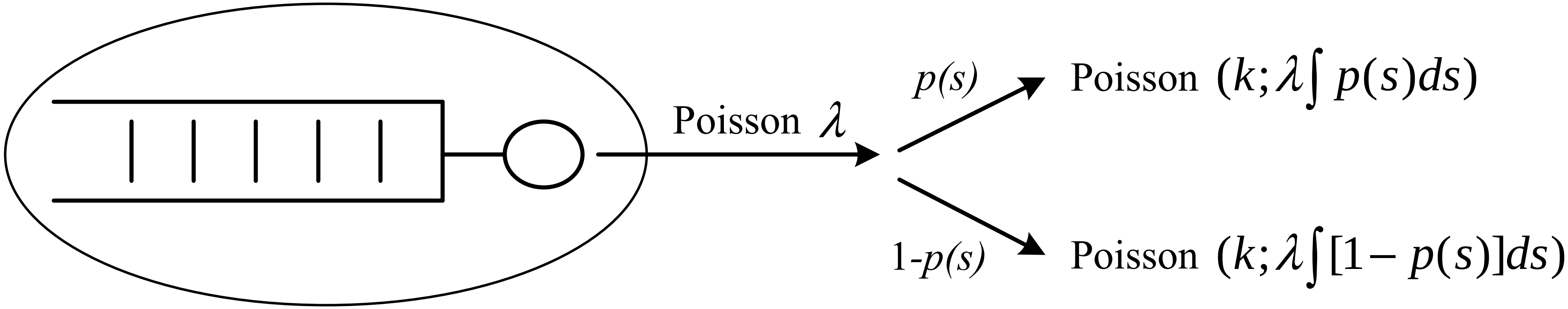
Theorem.

- Given a Poisson process $N = \{\tilde{n}(t), t \geq 0\}$;
- If $\tilde{n}_i(t)$ represents the number of type-i events that occur by time $t, i = 1, 2$;
- Arrival occurring at time s is a type-1 arrival with probability $p(s)$, and type-2 arrival with probability $1 - p(s)$

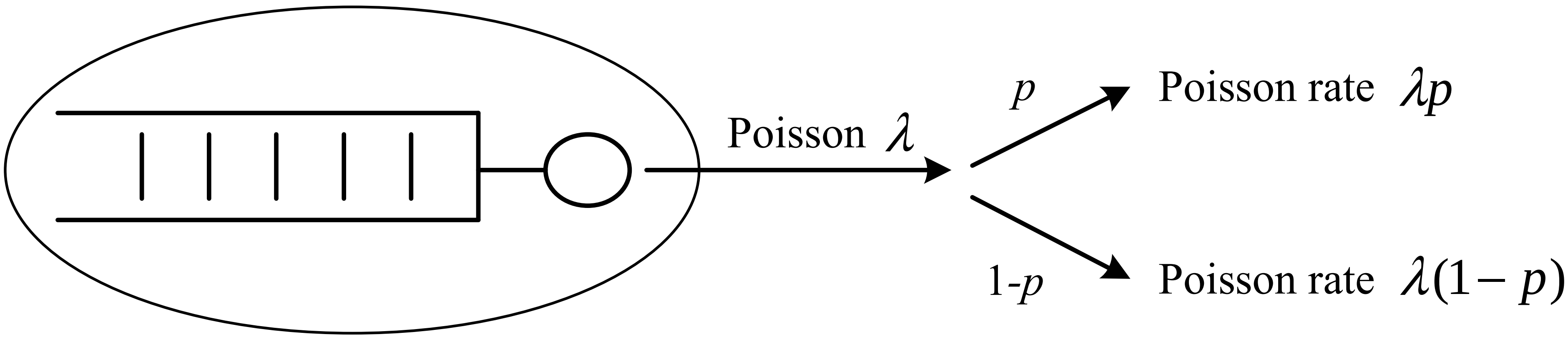
$\Downarrow$then

- $\tilde{n}_1, \tilde{n}_2$ are independent,
- $\tilde{n}_1(t) \sim P(k; \lambda tp)$, and
- $\tilde{n}_2(t) \sim P(k; \lambda t(1-p))$, where $p = \dfrac{1}{t}\displaystyle\int_0^t p(s)ds$

Decomposition of a Poisson Process



special case: If $p(s) = p$ is constant, then

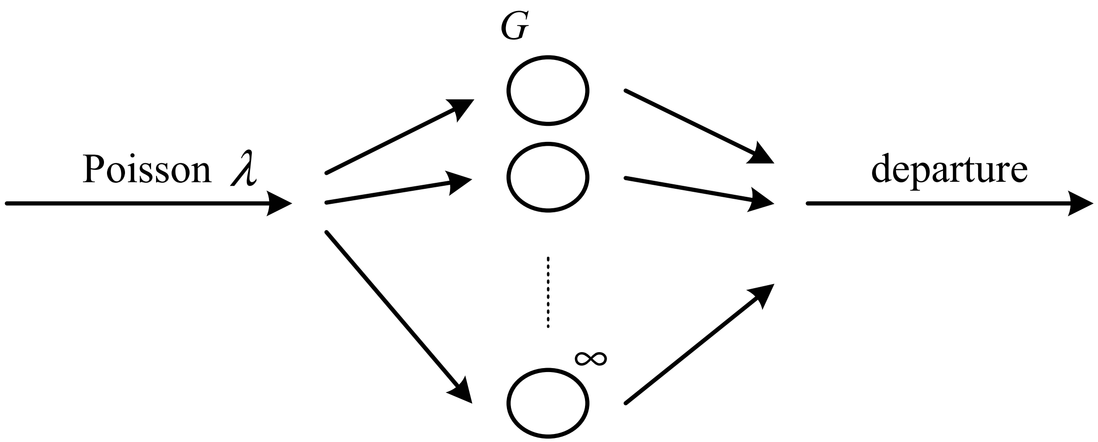


Decomposition of a Poisson Process

Proof.

Decomposition of a Poisson Process

Example (An Infinite Server Queue, textbook [Ross]).



- $G_{\tilde{s}}(t) = P(\tilde{S} \le t)$, where $\tilde{S}$ = service time
- $G_{\tilde{s}}(t)$ is independent of each other and of the arrival process
- $\tilde{n}_1(t)$: the number of customers which have left before t;
- $\tilde{n}_2(t)$: the number of customers which are still in the system at time t;

  $\Rightarrow \tilde{n}_1(t) \sim ?$ and $\tilde{n}_2(t) \sim ?$

Decomposition of a Poisson Process

Answer.

Non-homogeneous Poisson Processes

- The counting process $N = \{\tilde{n}(t), t \geq 0\}$ is said to be a *non-stationary* or *non-homogeneous* Poisson Process with time-varying intensity function $\lambda(t), t \geq 0$, if:
  1. $\tilde{n}(0) = 0$
  2. N has independent increments
  3. $P[\tilde{n}(t+h) - \tilde{n}(t) \geq 2] = o(h)$
  4. $P[\tilde{n}(t+h) - \tilde{n}(t) = 1] = \lambda(t) \cdot h + o(h)$
- Define "integrated intensity function" $m(t) = \int_0^t \lambda(t')dt'$.

  Theorem.

  $$P[\tilde{n}(t+s) - \tilde{n}(t) = n] = \frac{e^{-[m(t+s)-m(t)]}[m(t+s) - m(t)]^n}{n!}$$

  Proof. < Homework >.

Non-homogeneous Poisson Processes

Example. The "output process" of the $M/G/\infty$ queue is a non-homogeneous Poisson process having intensity function $\lambda(t) = \lambda G(t)$, where G is the service distribution.

Hint. Let $D(s, s + r)$ denote the number of service completions in the interval $(s, s + r]$ in $(0, t]$. If we can show that

- $D(s, s + r)$ follows a Poisson distribution with mean $\lambda \int_s^{s+r} G(y)dy$, and
- the numbers of service completions in disjoint intervals are independent,

then we are finished by definition of a non-homogeneous Poisson process.

Non-homogeneous Poisson Processes

Answer.

Non-homogeneous Poisson Processes

- Because of
  - the independent increment assumption of the Poisson arrival process, and
  - the fact that there are always servers available for arrivals,

$\Rightarrow$ the departure process has independent increments

Compound Poisson Processes

- A stochastic process $\{\tilde{x}(t), t \geq 0\}$ is said to be a *compound Poisson process* if
  - it can be represented as

$$\tilde{x}(t) = \sum_{i=1}^{\tilde{n}(t)} \tilde{y}_i, \quad t \geq 0$$

  - $\{\tilde{n}(t), t \geq 0\}$ is a Poisson process
  - $\{\tilde{y}_i, i \geq 1\}$ is a family of independent and identically distributed random variables which are also independent of $\{\tilde{n}(t), t \geq 0\}$
- The random variable $\tilde{x}(t)$ is said to be a compound Poisson random variable.
- $E[\tilde{x}(t)] =$ and $Var[\tilde{x}(t)] =$.

Compound Poisson Processes

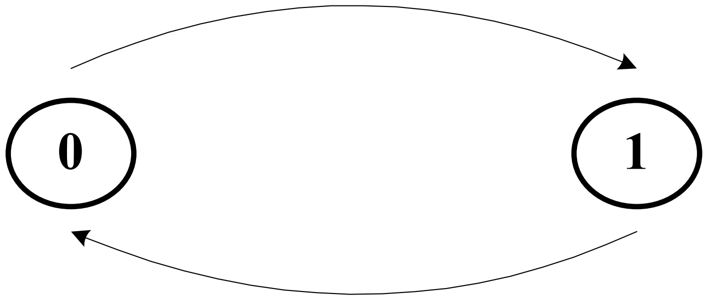
- **Example** (Batch Arrival Process). Consider a parallel-processing system where each job arrival consists of a possibly random number of tasks. Then we can model the arrival process as a compound Poisson process, which is also called a *batch arrival process*.

- Let $\tilde{y}_i$ be a random variable that denotes the number of tasks comprising a job. We derive the probability generating function $P_{\tilde{x}(t)}(z)$ as follows:

$$
\begin{aligned}
P_{\tilde{x}(t)}(z) &= E\left[z^{\tilde{x}(t)}\right] = E\left[E\left[\qquad\qquad\qquad\qquad\right]\right] = E\left[E\left[z^{\tilde{y}_1+\cdots+\tilde{y}_{\tilde{n}(t)}}\middle|\tilde{n}(t)\right]\right] \\
&= E\left[E\left[z^{\tilde{y}_1+\cdots+\tilde{y}_{\tilde{n}(t)}}\right]\right] \quad \text{(by independence of } \tilde{n}(t) \text{ and } \{\tilde{y}_i\}) \\
&= E\left[E\left[z^{\tilde{y}_1}\right]\cdots E\left[z^{\tilde{y}_{\tilde{n}(t)}}\right]\right] \quad \text{(by independence of } \tilde{y}_1, \cdots, \tilde{y}_{\tilde{n}(t)}) \\
&= E\left[\qquad\qquad\qquad\qquad\right] = P_{\tilde{n}(t)}\left(P_{\tilde{y}}(z)\right)
\end{aligned}
$$

Modulated Poisson Processes

- Assume that there are two states, 0 and 1, for a "modulating process."



- When the state of the modulating process equals 0 then the arrive rate of customers is given by λ_0, and when it equals 1 then the arrival rate is λ_1.
- The residence time in a particular modulating state is exponentially distributed with parameter μ and, after expiration of this time, the modulating process changes state.
- The initial state of the modulating process is randomly selected and is equally likely to be state 0 or 1.

Modulated Poisson Processes

- For a given period of time $(0, t)$, let Υ be a random variable that indicates the total amount of time that the modulating process has been in state 0. Let $\tilde{x}(t)$ be the number of arrivals in $(0, t)$.
- Then, given Υ, the value of $\tilde{x}(t)$ is distributed as a non-homogeneous Poisson process and thus

$$P[\tilde{x}(t) = n | \Upsilon = \tau] =$$

- As $\mu \to 0$, the probability that the modulating process makes no transitions within t seconds converges to 1, and we expect for this case that

$$P[\tilde{x}(t) = n] =$$

Modulated Poisson Processes

- As $\mu \to \infty$, then the modulating process makes an infinite number of transitions within t seconds, and we expect for this case that

$$P[\tilde{x}(t) = n] = \qquad\qquad\qquad\qquad, \quad \text{where } \beta = \frac{\lambda_0 + \lambda_1}{2}$$

- **Example** (Modeling Voice).
  - A basic feature of speech is that it comprises an alternation of silent periods and non-silent periods.
  - The arrival rate of packets during a talk spurt period is Poisson with rate λ_1 and silent periods produce a Poisson rate with $\lambda_0 \approx 0$.
  - The duration of times for talk and silent periods are exponentially distributed with parameters μ_1 and μ_0, respectively.

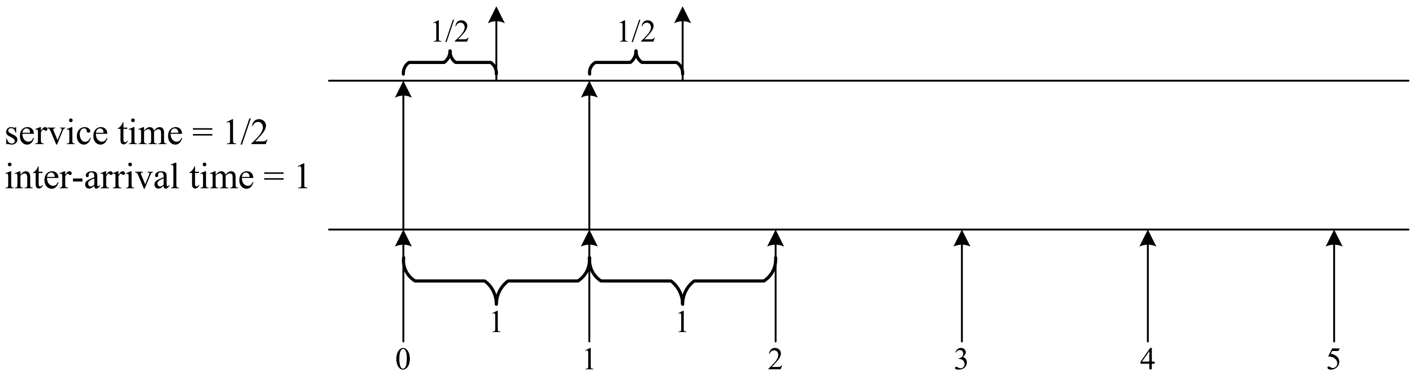
$\Rightarrow$ The model of the arrival stream of packets is given by a modulated Poisson process.

Poisson Arrivals See Time Averages (PASTA)

- PASTA says: as $t \to \infty$

$$
\begin{aligned}
& \text{Fraction of arrivals who see the system in a given state} \\
& \qquad\qquad\qquad \text{upon arrival (arrival average)} \\
= \; & \text{Fraction of time the system is in a given state (time average)} \\
= \; & \text{The system is in the given state at any random time} \\
& \qquad\qquad\qquad \text{after being steady}
\end{aligned}
$$

- Counter-example (textbook [Kao]: Example 2.7.1)



Poisson Arrivals See Time Averages (PASTA)

- Arrival average that an arrival will see an idle system =
- Time average of system being idle =

- Mathematically,
  - Let $X = \{\tilde{x}(t), t \geq 0\}$ be a stochastic process with state space S, and $B \subset S$
  - Define an indicator random variable

$$\tilde{u}(t) = \begin{cases} 1, & \text{if } \tilde{x}(t) \in B \\ 0, & \text{otherwise} \end{cases}$$

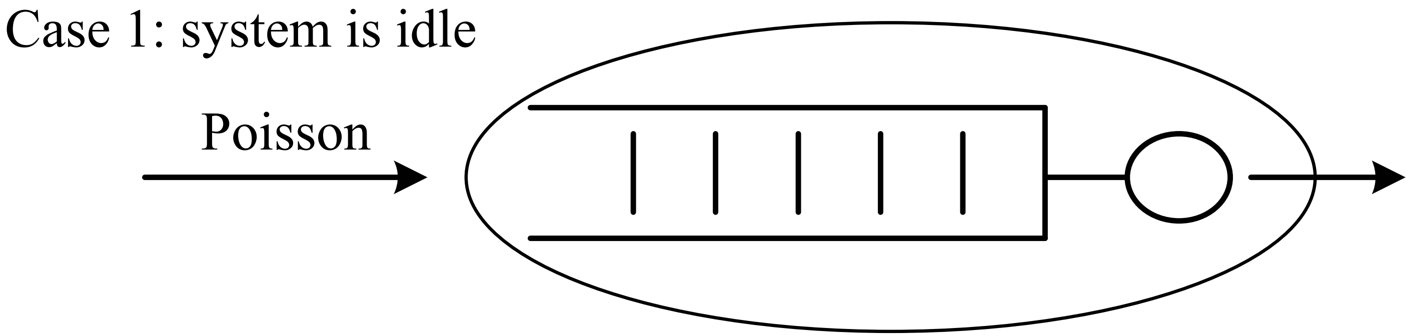
  - Let $N = \{\tilde{n}(t), t \geq 0\}$ be a Poisson process with rate λ denoting the arrival process

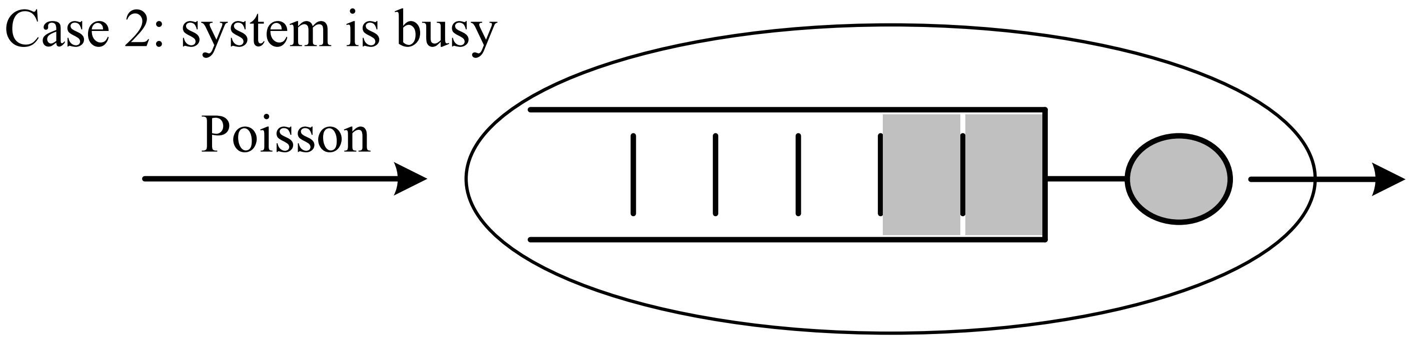
Poisson Arrivals See Time Averages (PASTA)

then,

- Condition – For PASTA to hold, we need the *lack of anticipation assumption* (LAA): for each $t \geq 0$,
  - the arrival process $\{\tilde{n}(t+u) - \tilde{n}(t), u \geq 0\}$ is independent of $\{\tilde{x}(s), 0 \leq s \leq t\}$ and $\{\tilde{n}(s), 0 \leq s \leq t\}$.
- Application:
  - To find the waiting time distribution of any arriving customer
  - Given: P[system is idle] $= 1 - \rho$; P[system is busy] $= \rho$

Poisson Arrivals See Time Averages (PASTA)





$$
\begin{aligned}
\Rightarrow P(\tilde{w} \le t) &= P(\tilde{w} \le t | \text{idle}) \cdot P(\text{idle upon arrival}) \\
&+ P(\tilde{w} \le t | \text{busy}) \cdot P(\text{busy upon arrival})
\end{aligned}
$$

Memoryless Property of the Exponential Distribution

- A random variable $\tilde{x}$ is said to be without memory, or *memoryless*, if

$$P[\tilde{x} > s + t | \tilde{x} > t] = P[\tilde{x} > s] \quad \text{for all } s, t \geq 0 \tag{3}$$

- The condition in Equation (3) is equivalent to

$$\frac{P[\tilde{x} > s + t, \tilde{x} > t]}{P[\tilde{x} > t]} = P[\tilde{x} > s]$$

or

$$P[\tilde{x} > s + t] = P[\tilde{x} > s]P[\tilde{x} > t] \tag{4}$$

- Since Equation (4) is satisfied when $\tilde{x}$ is exponentially distributed (for $e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$), it follows that exponential random variable are memoryless.
- Not only is the exponential distribution "memoryless," but it is the unique continuous distribution possessing this property.

Comparison of Two Exponential Random Variables

Suppose that $\tilde{x}_1$ and $\tilde{x}_2$ are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$. What is $P[\tilde{x}_1 < \tilde{x}_2]$?

$$
\begin{aligned}
P[\tilde{x}_1 < \tilde{x}_2] &= \int_0^\infty P[\tilde{x}_1 < \tilde{x}_2 | \tilde{x}_1 = x] \lambda_1 e^{-\lambda_1 x} dx \\
&= \int_0^\infty P[x < \tilde{x}_2] \lambda_1 e^{-\lambda_1 x} dx \\
&= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx \\
&= \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2) x} dx \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}
$$

Minimum of Exponential Random Variables

Suppose that $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n$ are independent exponential random variables, with $\tilde{x}_i$ having rate $\mu_i, i = 1, \cdots, n$. It turns out that the smallest of the $\tilde{x}_i$ is exponential with a rate equal to the sum of the μ_i.

$$\begin{aligned} P[\min(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) > x] &= P[\tilde{x}_i > x \text{ for each } i = 1, \cdots, n] \\ &= \prod_{i=1}^{n} P[\tilde{x}_i > x] \quad \text{(by independence)} \\ &= \prod_{i=1}^{n} e^{-\mu_i x} \\ &= exp\left\{-\left(\sum_{i=1}^{n} \mu_i\right) x\right\} \end{aligned}$$

How about $\max(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$? (exercise)