Chapter 4. Markov Chains

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Introduction

A *Markov chain*:

- Consider a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$ that takes on a finite or countable number of possible values denoted by the set of nonnegative integers $\{0, 1, 2, \ldots\}$.
- If $X_n = i$, then the process is said to be in state i at time n.
- Suppose that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j. That is,

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij}$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j$ and all $n \geq 0$.

$\Rightarrow$

Introduction

- *Markovian* property: For a Markov chain, the conditional distribution of any future state X_{n+1}, given the past states $X_0, X_1, \ldots, X_{n-1}$ and the present state X_n, is independent of the past states and depends only on the present state.
- The value P_{ij} represents the probability that the process will, when in state i, next make a transition into state j.
- Since probabilities are nonnegative and since the process must make a transition into some state, we have that

$$P_{ij} \geq 0, \quad i, j \geq 0; \quad \sum_{j=0}^{\infty} P_{ij} = 1, \quad i = 0, 1, \ldots$$

- Let P denote the matrix of one-step transition probabilities P_{ij}, so

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that

$$P = \left| \begin{array}{llll} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & & & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & & \vdots \end{array} \right|$$

Example 1. The $M/G/1$ Queue

- **The $M/G/1$ Queue.**
  - The customers arrive at a service center in accordance with a Poisson process with rate λ.
  - There is a single server and those arrivals finding the server free go immediately into service; all others wait in line until their service turn.
  - The service times of successive customers are assumed to be independent random variables having a common distribution G;
  - The service times are also assumed to be independent of the arrival process.
- If we let $X(t)$ denote the number of customers in the system at t, then $\{X(t), t \geq 0\}$ would not possess the Markovian property that the conditional distribution of the future depends only on the present and not on the past.

Example 1. The $M/G/1$ Queue

- For if we knew the number in the system at time t, then to predict future behavior,
  - we would "**not**" care how much time had elapsed since the last arrival (since the arrival process is memoryless)
  - we would care how long the person in service had already been there (since the service distribution G is arbitrary and therefore not memoryless)
- Let us only look at the system at "moments" when customers "depart".
  - let X_n denote the number of customers left behind by the nth departure, $n \geq 1$
  - let Y_n denote the number of customers arriving during the service period of the $(n+1)$st customer

Example 1. The $M/G/1$ Queue

- When $X_n > 0$, the nth departure leaves behind X_n customers — of which one enters service and the other $X_n - 1$ wait in line.
- Hence, at the next departure the system will contain the $X_n - 1$ customers that were in line in addition to any arrivals during the service time of the $(n+1)$st customer. Since a similar argument holds when $X_n = 0$, we see that

$$X_{n+1} =$$

- Since $Y_n, n \geq 1$, represent the number of arrivals in nonoverlapping service intervals, it follows, the arrival process being a Poisson process,

Example 1. The $M/G/1$ Queue

that they are independent and

$$P\{Y_n = j\} = \int_0^\infty$$

- From the above, it follows that $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain with transition probabilities given by

Example 2. The $G/M/1$ Queue

- **The $G/M/1$ Queue.**
  - The customers arrive at a single-server service center in accordance with an arbitrary renewal process having interarrival distribution G.
  - The service distribution is exponential with rate μ
- If we let X_n denote the number of customers in the system as seen by the nth arrival, it is easy to see that the process $\{X_n, n \geq 1\}$ is a Markov chain.
- Note that as long as there are customers to be served, the number of services in any length of time t is a Poisson random variable with mean μt. Therefore,

$$P_{i,i+1-j} = \int_0^\infty$$

Example 2. The $G/M/1$ Queue

- The above equation follows since if an arrival finds i in the system, then the next arrival will find $i+1$ minus the number served, and the probability that j will be served is easily seen (by conditioning on the time between the successive arrivals) to equal the right-hand side.
- The formula for P_{i0} is little different (it is the probability that at least $i+1$ Poisson events occur in a random length of time having distribution G) and thus is given by

$$P_{i0} = \int_0^\infty$$

- **Remark:** Note that in the previous two examples we were able to discover an *embedded* Markov chain by looking at the process only at certain time points, and by choosing these time points so as to exploit the lack of memory of the exponential distribution.

Example 3. The General Random Walk

The general random walk: sums of independent, identically distributed random variables.

- Let $X_i, i \geq 1$, be independent and identically distributed with

$$P\{X_i = j\} = a_j, \qquad j = 0, \pm 1, \ldots$$

- If we let

$$S_0 = 0 \text{ and } S_n = \sum_{i=1}^{n} X_i$$

  then $\{S_n, n \geq 0\}$ is a Markov chain for which

$$P_{ij} = a_{j-i}$$

- $\{S_n, n \geq 0\}$ is called the general random walk.

Example 4. The Simple Random Walk

- The random walk $\{S_n, n \geq 1\}$, where $S_n = \sum_1^n X_i$, is said to be a *simple random walk* if for some p, $0 < p < 1$,

$$\begin{aligned} P\{X_i = 1\} &= p \\ P\{X_i = -1\} &= q \equiv 1 - p \end{aligned}$$

  Thus in the simple random walk the process always either goes up one step (with probability p) or down one step (with probability q).

- Consider $|S_n|$, the absolute value of the simple random walk. The process $\{|S_n|, n \geq 1\}$ measures at each time unit the absolute distance of the simple random walk from the origin.

- To prove $\{|S_n|\}$ is itself a Markov chain, we first show that if $|S_n| = i$, then no matter what its previous values the probability that S_n equals i (as opposed to $-i$) is $p^i/(p^i + q^i)$.

Example 4. The Simple Random Walk

Proposition. If $\{S_n, n \geq 1\}$ is a simple random walk, then

$$P\{S_n = i | |S_n| = i, |S_{n-1}| = i_{n-1}, \ldots, |S_1| = i_1\} = \frac{p^i}{p^i + q^i}$$

Proof.

Example 4. The Simple Random Walk

- From the proposition, it follows upon conditioning on whether $S_n = +i$ or $-i$ that

$$P\{|S_{n+1}| = i + 1 | |S_n| = i, \ldots, |S_1|\}$$

$$=$$

$$=$$

- Hence, $\{|S_n|, n \geq 1\}$ is a Markov chain with transition probabilities

Chapman-Kolmogorov Equations

- P_{ij}: the one-step transition probabilities
- Define the n-step transition probabilities P_{ij}^n to be the probability that a process in state i will be in state j after n additional transitions. That is,

  $$P_{ij}^n = P\{X_{n+m} = j | X_m = i\}, \quad n \geq 0, \quad i, j \geq 0$$

  where, of course, $P_{ij}^1 = P_{ij}$.

- The *Chapman-Kolmogorov equations* provide a method for computing these n-step transition probabilities. These equations are

  $$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \quad \text{for all } n, m \geq 0, \quad \text{all } i, j,$$

  and are established by observing that

Chapman-Kolmogorov Equations

$$
\begin{aligned}
P_{ij}^{n+m} &= P\{X_{n+m} = j | X_0 = i\} \\
&= \\
&= \\
&=
\end{aligned}
$$

- Let $P^{(n)}$ denote the matrix of n-step transition probabilities P_{ij}^n, then the Chapman-Kolmogorov equations assert that

  $$P^{(n+m)} = P^{(n)} \cdot P^{(m)},$$

  where the dot represents matrix multiplication.

Chapman-Kolmogorov Equations

- Hence,

$$P^{(n)} = P \cdot P^{(n-1)} = P \cdot P \cdot P^{(n-2)} = \cdots = P^n,$$

  and thus $P^{(n)}$ may be calculated by multiplying the matrix P by itself n times.

Classification of States

- State j is said to be *accessible* from state i if for some $n \geq 0, P_{ij}^n > 0$.
- Two states i and j accessible to each other are said to *communicate*, and is denoted by $i \leftrightarrow j$.
- **Proposition.** Communication is an equivalence relation. That is:
    1. $i \leftrightarrow i$;
    2. if $i \leftrightarrow j$, then $j \leftrightarrow i$;
    3. if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

  Proof. The first two parts follow trivially from the definition of communication. To prove 3., suppose that $i \leftrightarrow j$ and $j \leftrightarrow k$; then there exists m, n such that $P_{ij}^m > 0$, $P_{jk}^n > 0$. Hence,

$$P_{ik}^{m+n} = \sum_{r=0}^{\infty} P_{ir}^m P_{rk}^n \geq P_{ij}^m P_{jk}^n > 0.$$

  Similarly, we may show there exists an s for which $P_{ki}^s > 0$.

Classification of States

- Two states that communicate are said to be in the same *class*; and by the above proposition, any two classes are either disjoint or identical.
- We say that the Markov chain is *irreducible* if there is only one class — that is, if all states communicate with each other.
- State i is said to have period d if $P_{ii}^{n} = 0$ whenever n is not divisible by d and d is the greatest integer with this property. (If $P_{ii}^{n} = 0$ for all $n > 0$, then define the period of i to be infinite.)
- A state with period 1 is said to be *aperiodic*.
- Let $d(i)$ denote the period of i. We now show by the following proposition that periodicity is a class property.
- **Proposition.** If $i \leftrightarrow j$, then $d(i) = d(j)$.

Classification of States

Proof.

- Let m and n be such that $P_{ij}^{m}P_{ji}^{n} > 0$, and suppose that $P_{ii}^{s} > 0$. Then

$$P_{jj}^{n+m} \geq P_{ji}^{n}P_{ij}^{m} > 0 \text{ and } P_{jj}^{n+s+m} \geq P_{ji}^{n}P_{ii}^{s}P_{ij}^{m} > 0.$$

- The second inequality follows, for instance, since the left-hand side represents the probability that starting in j the chain will be back in j after $n+s+m$ transitions, whereas the right-hand side is the probability of the same event subject to the further restriction that the chain is in i both after n and $n+s$ transitions.
- Hence, $d(j)$ divides both $n+m$ and $n+s+m$; thus $n+s+m-(n+m)=s$, whenever $P_{ii}^{s} > 0$. Therefore, $d(j)$ divides $d(i)$. A similar argument yields that $d(i)$ divides $d(j)$, thus $d(i)=d(j)$.

Classification of States

- For any states i and j, define f_{ij}^n to be the probability that, starting in i, the first transition into j occurs at time n.
- Formally,

$$\begin{aligned} f_{ij}^0 &= 0, \\ f_{ij}^n &= P\{X_n = j, X_k \neq j, k = 1, \ldots, n-1 | X_0 = i\}. \end{aligned}$$

- Let

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n.$$

  Then f_{ij} denotes the probability of ever making a transition into state j, given that the process starts in i. (Note that for $i \neq j$, f_{ij} is positive if, and only if, j is accessible from i.)
- State j is said to be *recurrent* if $f_{jj} = 1$, and *transient* otherwise.

Classification of States

Proposition. State j is recurrent if, and only if,

$$\sum_{n=1}^{\infty} P_{jj}^{n} = \infty.$$

Proof.

- State j is recurrent if, with probability 1, a process starting at j will eventually return.
- However, by the Markovian property it follows that the process probabilistically restarts itself upon returning to j. Hence, with probability 1, it will return again to j.
- Repeating this argument, we see that, with probability 1, the number of visits to j will be infinite and will thus have infinite expectation.
- On the other hand, suppose j is transient. Then each time the

Classification of States

process returns to j there is a positive probability $1 - f_{jj}$ that it will never again return; hence the number of visits is geometric with finite mean $1/(1 - f_{jj})$.

- By the above argument we see that state j is recurrent if, and only if,

$$E[\text{number of visits to } j | X_0 = j] = \infty.$$

- But, letting $I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{otherwise,} \end{cases}$

  it follows that $\sum_0^\infty I_n$ denotes the number of visits to j. Since

$$E[\sum_{n=0}^{\infty} I_n | X_0 = j] = \sum_{n=0}^{\infty} E[I_n | X_0 = j] = \sum_{n=0}^{\infty} P_{jj}^n,$$

  the result follows.

Classification of States

- The proposition also shows that a *transient* state will only be visited a finite number of times (hence the name transient).
- This leads to the conclusion that in a *finite-state* Markov chain not all states can be transient.
- To see this, suppose the states are $0, 1, \ldots, M$ and suppose that they are all transient. Then after a finite amount of time (say after time T_0) state 0 will never be visited, and after a time (say T_1) state 1 will never be visited, and after a time (say T_2) state 2 will never be visited, and so on.
- Thus, after a finite time $T = \max\{T_0, T_1, \ldots, T_M\}$ no states will be visited. But as the process must be in some state after time T, we arrive at a contradiction, which shows that at least one of the states must be recurrent.

Classification of States

We use the above proposition to prove that *recurrence*, like periodicity, is a class property.

Corollary. If i is recurrent and $i \leftrightarrow j$, then j is recurrent.

Proof. Let m and n be such that $P_{ij}^{n} > 0$, $P_{ji}^{m} > 0$. Now for any $s \geq 0$

$$P_{jj}^{m+n+s} \geq P_{ji}^{m} P_{ii}^{s} P_{ij}^{n}$$

and thus

$$\sum_{s} P_{jj}^{m+n+s} \geq P_{ji}^{m} P_{ij}^{n} \sum_{s} P_{ii}^{s} = \infty,$$

and the result follows from the above proposition.

Classification of States

Example. The Simple Random Walk.

- The Markov chain whose state space is the set of all integers and has transition probabilities

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 0, \pm 1, \ldots,$$

  where $0 < p < 1$, is called the simple random walk.

- One interpretation of this process is that it represents the winnings of a gambler who on each play of the game either wins or loses one dollar.
- Since all states clearly communicate it follows from the corollary that they are either all transient or all recurrent.
- Consider state 0 and attempt to determine if $\sum_{n=1}^{\infty} P_{00}^n$ is finite or infinite.
- Since it is impossible to be even (using the gambling model

Classification of States

interpretation) after an odd number of plays, we must, of course, have that

$$P_{00}^{2n+1} = 0, \qquad n = 1, 2, \ldots$$

- On the other hand, the gambler would be even after $2n$ trials if, and only if, he won n of these and lost n of these.
- As each play of the game results in a win with probability p and a loss with probability $1 - p$, the desired probability is thus the binomial probability

$$P_{00}^{2n} = (\,{}^{2n}_{\;n}\,)p^n(1-p)^n = \frac{(2n)!}{n!n!}(p(1-p))^n, \qquad n = 1, 2, 3, \ldots$$

- By using an approximation, due to Stirling, which asserts that

$$n! \sim n^{n+1/2}e^{-n}\sqrt{2\pi}$$

Classification of States

where we say that $a_n \sim b_n$ when $\lim_{n\to\infty}(a_n/b_n) = 1$, we obtain

$$P_{00}^{2n} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}.$$

- It is easy to verify that if $a_n \sim b_n$, then $\sum_n a_n < \infty$, if, and only if, $\sum_n b_n < \infty$. Hence $\sum_{n=1}^{\infty} P_{00}^n$ will converge if, and only if,

$$\sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{\sqrt{\pi n}}$$

  does.
- However, $4p(1-p) \leq 1$ with equality holding if, and only if, $p = \frac{1}{2}$. Hence, $\sum_{n=1}^{\infty} P_{00}^n = \infty$ if, and only if, $p = \frac{1}{2}$. Thus, the chain is recurrent when $p = \frac{1}{2}$ and transient if $p \neq \frac{1}{2}$.

Classification of States

Remark.

- When $p = \frac{1}{2}$, the above process is called a *symmetric random walk*. We could also look at symmetric random walks in more than one dimension.
- For instance, in the two-dimensional symmetric random walk the process would, at each transition, either take one step to the left, right, up, or down, each having probability $\frac{1}{4}$.
- Similarly, in three dimensions the process would, with probability $\frac{1}{6}$, make a transition to any of the six adjacent points.
- By using the same method as in the one-dimensional random walk it can be shown that the two-dimensional symmetric random walk is recurrent, but all higher-dimensional random walks are transient.

Classification of States

Corollary. If $i \leftrightarrow j$ and j is recurrent, then $f_{ij} = 1$.

Proof.

- Suppose $X_0 = i$, and let n be such that $P_{ij}^n > 0$.
- Say that we miss opportunity 1 if $X_n \neq j$. If we miss opportunity 1, then let T_1 denote the next time we enter i (T_1 is finite with probability 1 by the previous corollary).
- Say that we miss opportunity 2 if $X_{T_1+n} \neq j$. If opportunity 2 is missed, let T_2 denote the next time we enter i and say that we miss opportunity 3 if $X_{T_2+n} \neq j$, and so on.
- It is easy to see that the opportunity number of the first success is a geometric random variable with mean $1/P_{ij}^n$, and is thus finite with probability 1. The result follows since i being recurrent implies that the number of potential opportunities is infinite.

Classification of States

Remark.

- Let $N_j(t)$ denote the number of transitions into j by time t.
- If j is recurrent and $X_0 = j$, then as the process probabilistically starts over upon transitions into j, it follows that $\{N_j(t), t \geq 0\}$ is a renewal process with interarrival distribution $\{f_{jj}^n, n \geq 1\}$.
- If $X_0 = i, i \leftrightarrow j$, and j is recurrent, then $\{N_j(t), t \geq 0\}$ is a delayed renewal process with initial interarrival distribution $\{f_{ij}^n, n \geq 1\}$.

Limit Theorems

- It is easy to show that if state j is transient, then

  $$\sum_{n=1}^{\infty} P_{ij}^{n} < \infty \qquad \text{for all } i,$$

  meaning that, starting in i, the expected number of transitions into state j is finite. As a consequence it follows that for j transient $P_{ij}^{n} \to 0$ as $n \to \infty$.

- Let μ_{jj} denote the expected number of transitions needed to return to state j. That is,

  $$\mu_{jj} = \begin{cases} \infty & \text{if } j \text{ is transient} \\ \sum_{n=1}^{\infty} n f_{jj}^{n} & \text{if } j \text{ is recurrent} \end{cases}$$

Limit Theorems

- By interpreting transitions into state j as being renewals, we obtain the following theorem from Chapter 3.
- **Theorem.** If i and j communicate, then:

  1. $P\{\lim_{t\to\infty} N_j(t)/t = 1/\mu_{jj} | X_0 = i\} = 1$
  2. $\lim_{n\to\infty} \sum_{k=1}^{n} P_{ij}^{k}/n = 1/\mu_{jj}$
  3. If j is *aperiodic*, then $\lim_{n\to\infty} P_{ij}^{n} = 1/\mu_{jj}$
  4. If j has *period* d, then $\lim_{n\to\infty} P_{jj}^{nd} = d/\mu_{jj}$

Limit Theorems

- If state j is recurrent, then we say that it is *positive* recurrent if $\mu_{jj} < \infty$ and *null* recurrent if $\mu_{jj} = \infty$.
- If we let

$$\pi_j = \lim_{n \to \infty} P_{jj}^{nd(j)},$$

it follows that a recurrent state j is positive recurrent if $\pi_j > 0$ and null recurrent if $\pi_j = 0$.

- **Proposition.** Positive (null) recurrence is a class property.
- A positive recurrent, aperiodic state is called *ergodic*.
- Before presenting a theorem that shows how to obtain the limiting probabilities in the ergodic case, we need the following definition.

Limit Theorems

- **Definition.** A probability distribution $\{P_j, j \geq 0\}$ is said to be *stationary* for the Markov chain if

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij},\ j \geq 0.$$

- If the probability distribution of X_0 — say $P_j = P\{X_0 = j\}, j \geq 0$ — is a stationary distribution, then

$$\begin{aligned} P\{X_1 = j\} &= \sum_{i=0}^{\infty} P\{X_1 = j | X_0 = i\} P\{X_0 = i\} \\ &= \sum_{i=0}^{\infty} P_i P_{ij} = P_j \end{aligned}$$

Limit Theorems

and, by induction,

$$\begin{aligned} P\{X_n = j\} &= \sum_{i=0}^{\infty} P\{X_n = j | X_{n-1} = i\} P\{X_{n-1} = i\} \\ &= \sum_{i=0}^{\infty} P_{ij} P_i = P_j. \end{aligned}$$

- Hence, if the initial probability distribution is the stationary distribution, then X_n will have the same distribution for all n.
- In fact, as $\{X_n, n \geq 0\}$ is a Markov chain, it easily follows from this that for each $m \geq 0, X_n, X_{n+1}, \ldots, X_{n+m}$ will have the same joint distribution for each n; in other words, $\{X_n, n \geq 0\}$ will be a stationary process.

Limit Theorems

Theorem. An *irreducible aperiodic Markov chain* belongs to one of the following two classes:

1. Either the states are *all transient* or *all null recurrent*; in this case, $P_{ij}^n \to 0$ as $n \to \infty$ for all i, j and there exists no stationary distribution.
2. Or else, all states are *positive recurrent*, that is,

$$\pi_j = \lim_{n\to\infty} P_{ij}^n > 0$$

   In this case, $\{\pi_j, j = 0, 1, 2, \ldots\}$ is a *stationary distribution* and there exists no other stationary distribution.

Limit Theorems

Proof. We will first prove 2. To begin, note that

$$P_{ij}^{n+1} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj} \geq \sum_{k=0}^{M} P_{ik}^{n} P_{kj} \quad \text{for all } M.$$

Letting $n \to \infty$ yields

$$\pi_j \geq \sum_{k=0}^{M} \pi_k P_{kj} \quad \text{for all } M,$$

implying that

$$\pi_j \geq \sum_{k=0}^{\infty} \pi_k P_{kj}, \quad j \geq 0.$$

To show that the above is actually an equality, suppose that the inequality is strict for some j. Then upon adding these inequalities we

Limit Theorems

obtain

$$\sum_{j=0}^{\infty} \pi_j > \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \pi_k P_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \pi_k,$$

which is a contradiction. Therefore,

$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}, \qquad j = 0, 1, 2, \ldots$$

Putting $P_j = \pi_j / \sum_0^\infty \pi_k$, we see that $\{P_j, j = 0, 1, 2, \ldots\}$ is a stationary distribution, and hence at least one stationary distribution exists.

Now let $\{P_j, j = 0, 1, 2, \ldots\}$ be any stationary distribution. Then if $\{P_j, j = 0, 1, 2, \ldots\}$ is the probability distribution of X_0, then

$$P_j = P\{X_n = j\}$$

Limit Theorems

$$= \sum_{i=0}^{\infty} P\{X_n = j | X_0 = i\} P\{X_0 = i\} = \sum_{i=0}^{\infty} P_{ij}^n P_i. \qquad (2)$$

From (2) we see that

$$P_j \geq \sum_{i=0}^{M} P_{ij}^n P_i \quad \text{for all } M.$$

Letting n and then M approach ∞ yields

$$P_j \geq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j.$$

To go the other way and show that $P_j \leq \pi_j$, use (2) and the fact that

Limit Theorems

$P_{ij}^n \leq 1$ to obtain

$$P_j \leq \sum_{i=0}^{M} P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_i \quad \text{for all } M,$$

and letting $n \to \infty$ gives

$$P_j \leq \sum_{i=0}^{M} \pi_j P_i + \sum_{i=M+1}^{\infty} P_i \quad \text{for all } M.$$

Since $\sum_0^\infty P_i = 1$, we obtain upon letting $M \to \infty$ that

$$P_j \leq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j.$$

Limit Theorems

If the states are transient or null recurrent and $\{P_j, j = 0, 1, 2, \ldots\}$ is a stationary distribution, then Equation (2) holds and $P_{ij}^n \to 0$, which is clearly impossible. Thus, for case 1., no stationary distribution exists and the proof is complete.

Limit Theorems

Remarks.

1. When the situation is as described in part (ii) of the above theorem we say that the Markov chain is *ergodic*.
2. If the process is started with the limiting probabilities, then the resultant Markov chain is stationary.
3. In the irreducible, positive recurrent, *periodic* case we still have that the $\pi_j, j \geq 0$, are the unique nonnegative solution of

$$\pi_j = \sum_i \pi_i P_{ij} \text{ and } \sum_j \pi_j = 1.$$

But now π_j must be interpreted as the *long-run proportion of time* that the Markov chain is in state j. Thus, $\pi_j = 1/\mu_{jj}$, whereas the

Limit Theorems

limiting probability of going from j to j in $nd(j)$ steps is given by

$$\lim_{n\to\infty} P_{jj}^{nd} = \frac{d}{\mu_{jj}} = d\pi_j$$

where d is the period of the Markov chain.

Limiting Prob. for the Embedded $M/G/1$ Queue

- Consider again the embedded Markov chain of the $M/G/1$ system and let

  $$a_j = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dG(x).$$

  That is, a_j is the probability of j arrivals during a service period. The transition probabilities for this chain are

  $$\begin{aligned} P_{0j} &= a_j, \\ P_{ij} &= a_{j-i+1}, \qquad i > 0,\ j \geq i-1 \\ P_{ij} &= 0, \qquad j < i-1 \end{aligned}$$

- Let $\rho = \sum_j j a_j$. Since ρ equals the mean number of arrivals during a service period, it follows that

  $$\rho = \lambda E[S],$$

Limiting Prob. for the Embedded $M/G/1$ Queue

where S is a service time having distribution G.

- We show that “the Markov chain is positive recurrent when $\rho < 1$” by solving the system of equations

$$\pi_j = \sum_i \pi_i P_{ij}.$$

- These equations take the form

$$\pi_j = \pi_0 a_j + \sum_{i=1}^{j+1} \pi_i a_{j-i+1}, \qquad j \geq 0. \tag{3}$$

To solve, we introduce the generating functions

$$\pi(s) = \sum_{j=0}^{\infty} \pi_j s^j, \qquad A(s) = \sum_{j=0}^{\infty} a_j s^j.$$

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Multiplying both sides of (3) by s^j and summing over j yields

$$\begin{aligned}\pi(s) &= \pi_0 A(s) + \sum_{j=0}^{\infty}\sum_{i=1}^{j+1}\pi_i a_{j-i+1}s^j \\ &= \pi_0 A(s) + s^{-1}\sum_{i=1}^{\infty}\pi_i s^i \sum_{j=i-1}^{\infty} a_{j-i+1}s^{j-i+1} \\ &= \pi_0 A(s) + (\pi(s)-\pi_0)A(s)/s,\end{aligned}$$

or

$$\pi(s) = \frac{(s-1)\pi_0 A(s)}{s-A(s)}.$$

To compute π_0 we let $s \to 1$ in the above.

- As

$$\lim_{s\to 1} A(s) = \sum_{i=0}^{\infty} a_i = 1,$$

Limiting Prob. for the Embedded $M/G/1$ Queue

this gives

$$\begin{aligned}\lim_{s\to 1}\pi(s) &= \pi_0 \lim_{s\to 1}\frac{s-1}{s-A(s)}\\ &= \pi_0(1-A'(1))^{-1},\end{aligned}$$

where the last equality follows from L'hospital's rule.

- Now

$$A'(1)=\sum_{i=0}^{\infty} i a_i = \rho,$$

and thus

$$\lim_{s\to 1}\pi(s)=\frac{\pi_0}{1-\rho}.$$

- However, since $\lim_{s\to 1}\pi(s)=\sum_{i=0}^{\infty}\pi_i$, this implies that $\sum_{i=0}^{\infty}\pi_i=\pi_0/(1-\rho)$; thus stationary probabilities exist if and only if

Limiting Prob. for the Embedded $M/G/1$ Queue

$\rho < 1$, and in this case,

$$\pi_0 = 1 - \rho = 1 - \lambda E[S].$$

Hence, when $\rho < 1$, or, equivalently, when $E[S] < 1/\lambda$,

$$\pi(s) = \frac{(1 - \lambda E[S])(s-1)A(s)}{s - A(s)}.$$

A Population Model

- Suppose that during each time period, every member of a population independently dies with probability p, and also that the number of new members that join the population in each time period is a Poisson random variable with mean λ.
- If we let X_n denote the number of members of the population at the beginning of period n, then it is easy to see that $\{X_n, n = 1, \ldots\}$ is a Markov chain.
- To find the stationary probabilities of this chain, suppose that X_0 is distributed as a Poisson random variable with parameter α.
- Since each of these X_0 individuals will independently be alive at the beginning of the next period with probability $1 - p$, it follows that the number of them that are still in the population at time 1 is a Poisson random variable with mean $\alpha(1 - p)$.

A Population Model

- As the number of new members that join the population by time 1 is an independent Poisson random variable with mean λ, it thus follows that X_1 is a Poisson random variable with mean $\alpha(1-p)+\lambda$. Hence, if

  $$\alpha = \alpha(1-p) + \lambda$$

  then the chain would be stationary.

- Hence, by the uniqueness of the stationary distribution, we can conclude that the stationary distribution is Poisson with mean λ/p. That is,

  $$\pi_j = e^{-\lambda/p}(\lambda/p)^j/j!, \qquad j = 0, 1, \ldots$$

Transitions Among Classes

We prove that a recurrent class is a *closed* class in the sense that once entered it is never left.

Proposition. Let R be a recurrent class of states. If $i \in R, j \notin R$, then $P_{ij} = 0$.

Proof. Suppose $P_{ij} > 0$. Then, as i and j do not communicate (since $j \notin R$), $P_{ji}^n = 0$ for all n. Hence if the process starts in state i, there is a positive probability of at least P_{ij} that the process will never return to i. This contradicts the fact that i is recurrent, and so $P_{ij} = 0$.

Transitions Among Classes

- Let j be a given recurrent state and let T denote the set of all transient states. For $i \in T$, we are often interested in computing f_{ij}, the probability of ever entering j given that the process starts in i.
- The following proposition, by conditioning on the state after the initial transition, yields a set of equations satisfied by the f_{ij}.
- **Proposition 4.4.2.** If j is recurrent, then the set of probabilities $\{f_{ij}, i \in T\}$ statisfies

$$f_{ij} = \sum_{k \in T} P_{ik} f_{kj} + \sum_{k \in R} P_{ik},\ i \in T$$

where R denotes the set of states communicating with j.

Transitions Among Classes

Proof.

$$\begin{aligned}
f_{ij} &= P\{N_j(\infty) > 0 | X_0 = i\} \\
&= \sum_{all\ k} P\{N_j(\infty) > 0 | X_0 = i, X_1 = k\} P\{X_1 = k | X_0 = i\} \\
&= \sum_{k \in T} f_{kj} P_{ik} + \sum_{k \in R} f_{kj} P_{ik} + \sum_{\substack{k \notin R \\ k \notin T}} f_{kj} P_{ik} \\
&= \sum_{k \in T} f_{kj} P_{ik} + \sum_{k \in R} P_{ik},
\end{aligned}$$

where we have used the corollary in asserting that $f_{kj} = 1$ for $k \in R$ and the proposition in asserting that $f_{kj} = 0$ for $k \notin T, k \notin R$.

The Gambler's Ruin Problem

- Consider a gambler who at each play of the game has probability p of winning 1 unit and probability $q = 1 - p$ of losing 1 unit.
- Assuming successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will $\underline{\text{reach } N \text{ before reaching } 0}$?
- If we let X_n denote the player's fortune at time n, then the process $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain with transition probabilities

$$\begin{aligned} P_{00} &= P_{NN} = 1, \\ P_{i,i+1} &= p = 1 - P_{i,i-1}, \qquad i = 1, 2, \ldots, N-1. \end{aligned}$$

- This Markov chain has three classes, namely, $\{0\}$, $\{1, 2, \ldots, N-1\}$, and $\{N\}$, the first and third class being recurrent and the second transient.

The Gambler's Ruin Problem

- Since each transient state is only visited finitely often, it follows that, after some finite amount of time, the gambler will either attain her goal of N or go broke.
- Let $f_i \equiv f_{iN}$ denote the probability that, starting with $i, 0 \leq i \leq N$, the gambler's fortune will eventually reach N.
- By conditioning on the outcome of the initial play of the game (or, equivalently, by using Proposition 4.4.2), we obtain

$$f_i = pf_{i+1} + qf_{i-1}, \qquad i = 1, 2, \ldots, N-1,$$

or, equivalently, since $p + q = 1$,

$$f_{i+1} - f_i = \frac{q}{p}(f_i - f_{i-1}), \qquad i = 1, 2, \ldots, N-1.$$

The Gambler's Ruin Problem

- Since $f_0 = 0$, we see from the above that

$$\begin{aligned}
f_2 - f_1 &= \frac{q}{p}(f_1 - f_0) = \frac{q}{p}f_1 \\
f_3 - f_2 &= \frac{q}{p}(f_2 - f_1) = (\frac{q}{p})^2 f_1 \\
&\vdots \\
f_i - f_{i-1} &= \frac{q}{p}(f_{i-1} - f_{i-2}) = (\frac{q}{p})^{i-1} f_1 \\
&\vdots \\
f_N - f_{N-1} &= (\frac{q}{p})(f_{N-1} - f_{N-2}) = (\frac{q}{p})^{N-1} f_1.
\end{aligned}$$

Adding the first $i-1$ of these equations yields

$$f_i - f_1 = f_1[(\frac{q}{p}) + (\frac{q}{p})^2 + \cdots + (\frac{q}{p})^{i-1}]$$

The Gambler's Ruin Problem

or

$$f_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)} f_1 & \text{if } \dfrac{q}{p} \neq 1 \\ i f_1 & \text{if } \dfrac{q}{p} = 1. \end{cases}$$

Using $f_N = 1$ yields

$$f_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq \frac{1}{2} \\ \dfrac{i}{N} & \text{if } p = \frac{1}{2}. \end{cases}$$

- It is interesting to note that as $N \to \infty$

$$f_i \to \begin{cases} 1-(q/p)^i & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \leq \frac{1}{2}. \end{cases}$$

The Gambler's Ruin Problem

- Hence, from the continuity property of probabilities, it follows that
  - if $p > \frac{1}{2}$, there is a positive probability that the gambler's fortune will converge to infinity;
  - whereas if $p \leq \frac{1}{2}$, then, with probability 1, the gambler will eventually go broke when playing against an infinitely rich adversary.

Time-Reversible Markov Chains

- An irreducible positive recurrent Markov chain is stationary if the initial state is chosen according to the stationary probabilities. (In the case of an ergodic chain this is equivalent to imagining that the process begins at time $t = -\infty$.)
- Consider now a stationary Markov chain having transition probabilities P_{ij} and stationary probabilities π_i, and suppose that starting at some time we trace the sequence of states going backwards in time.
- That is, starting at time n consider the sequence of states $X_n, X_{n-1}, \ldots$. It turns out that this sequence of states is itself a Markov chain with transition probabilities P^*_{ij} defined by

$$\begin{aligned} P^*_{ij} &= P\{X_m = j | X_{m+1} = i\} \\ &= \frac{P\{X_{m+1} = i | X_m = j\} P\{X_m = j\}}{P\{X_{m+1} = i\}} \end{aligned}$$

Time-Reversible Markov Chains

$$= \quad \frac{\pi_j P_{ji}}{\pi_i}.$$

- To prove that the reversed process is indeed a Markov chain we need to verify that

$$P\{X_m = j | X_{m+1} = i, X_{m+2}, X_{m+3}, \ldots\} = P\{X_m = j | X_{m+1} = i\}.$$

- Think of the present time as being time $m+1$. Then, since $X_n, n \geq 1$ is a Markov chain it follows that given the present state X_{m+1} the past state X_m and the future states $X_{m+2}, X_{m+3}, \ldots$ are independent. But this is exactly what the preceding equation states.
- Thus the reversed process is also a Markov chain with transition probabilities given by

$$P^*_{ij} = \frac{\pi_j P_{ji}}{\pi_i}.$$

Time-Reversible Markov Chains

- If $P^*_{ij} = P_{ij}$ for all i, j, then the Markov chain is said to be *time reversible.*
- The condition for time reversibility, namely, that

$$\pi_i P_{ij} = \pi_j P_{ji} \qquad \text{for all } i, j,$$

  can be interpreted as stating that, for all states i and j, the rate at which the process goes from i to j (namely, $\pi_i P_{ij}$) is equal to the rate at which it goes from j to i (namely, $\pi_j P_{ji}$).
- This is an obvious necessary condition for time reversibility since a transition from i to j going backward in time is equivalent to a transition from j to i going forward in time; that is, if $X_m = i$ and $X_{m-1} = j$, then a transition from i to j is observed if we are looking backward in time and one from j to i if we are looking forward in time.

Time-Reversible Markov Chains

Theorem. A stationary Markov chain is time reversible if, and only if, starting in state i, any path back to i has the same probability as the reversed path, for all i. That is, if

$$P_{i,i_1}P_{i_1,i_2}\dots P_{i_k,i} = P_{i,i_k}P_{i_k,i_{k-1}}\dots P_{i_1,i}$$

for all states $i, i_1, \dots, i_k$.

Proof. The proof of necessity is straightforward. To prove sufficiency fix states i and j and rewrite the above equation as

$$P_{i,i_1}P_{i_1,i_2}\dots P_{i_k,j}P_{ji} = P_{ij}P_{j,i_k}\dots P_{i_1,i}.$$

Summing the above over all states $i_1, i_2, \dots, i_k$ yields

$$P_{ij}^{k+1}P_{ji} = P_{ij}P_{ji}^{k+1}.$$

Time-Reversible Markov Chains

Hence

$$P_{ji} \frac{\sum_{k=1}^{n} P_{ij}^{k+1}}{n} = P_{ij} \frac{\sum_{k=1}^{n} P_{ji}^{k+1}}{n}.$$

Letting $n \to \infty$ now yields

$$P_{ji}\pi_j = P_{ij}\pi_i,$$

which establishes the result.

Time-Reversible Markov Chains

Theorem. Consider an irreducible Markov chain with transition probabilities P_{ij}. If one can find nonnegative numbers $\pi_i, i \geq 0$, summing to unity, and a transition probability matrix $P^* = [P^*_{ij}]$ such that

$$\pi_i P_{ij} = \pi_j P^*_{ji},$$

then the $\pi_i, i \geq 0$, are the stationary probabilities and P^*_{ij} are the transition probabilities of the reversed chain.

Proof. Summing the above equality over all i yields

$$\begin{aligned} \sum_i \pi_i P_{ij} &= \pi_j \sum_i P^*_{ji} \\ &= \pi_j. \end{aligned}$$

Hence, the π_i's are the stationary probabilities of the forward chain

Time-Reversible Markov Chains

(and also of the reversed chain; why?). Since

$$P_{ji}^* = \frac{\pi_i P_{ij}}{\pi_j},$$

it follows that the P_{ij}^* are the transition probabilities of the reversed chain.

Remarks. The importance of the previous two theorems is that we can sometimes guess at the nature of the reversed chain and then use the set of equations $\pi_i P_{ij} = \pi_j P_{ji}^*$ to obtain both the stationary probabilities and the P_{ij}^*. An example will be provided on the course web site for self-reading.

Semi-Markov Processes

- A *semi-Markov process* is one that changes states in accordance with a Markov chain but takes a random amount of time between changes.
- More specifically, consider a stochastic process with states $0, 1, \ldots$, which is such that, whenever it enters state $i, i \geq 0$:
  - The next state it will enter is state j with probability $P_{ij}, i, j \geq 0$.
  - Given that the next state to be entered is state j, the time until the transition from i to j occurs has distribution F_{ij}.

  If we let $Z(t)$ denote the state at time t, then $\{Z(t), t \geq 0\}$ is called a semi-Markov process.
- Thus a semi-Markov process does not possess the Markovian property that given the present state the future is independent of the past.
- In predicting the future not only would we want to know the present state, but also the length of time that has been spent in that state.

Semi-Markov Processes

- A Markov chain is a semi-Markov process in which

  $$F_{ij}(t) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1. \end{cases}$$

  That is, all transition times of a Markov chain are identically 1.

- Let H_i denote the distribution of time that the semi-Markov process spends in state i before making a transition. That is, by conditioning on the next state, we see

  $$H_i(t) = \sum_j P_{ij} F_{ij}(t),$$

  and let μ_i denote its mean. That is,

  $$\mu_i = \int_0^\infty x dH_i(x).$$

Semi-Markov Processes

- If we let X_n denote the nth state visited, then $\{X_n, n \geq 0\}$ is a Markov chain with transition probabilities P_{ij}. It is called the *embedded* Markov chain of the semi-Markov process. We say that the semi-Markov process is *irreducible* if the embedded Markov chain is irreducible as well.
- Let T_{ii} denote the time between successive transitions into state i and let $\mu_{ii} = E[T_{ii}]$. By using the theory of alternating renewal processes, we could derive an expression for the limiting probabilities of a semi-Markov process.

Semi-Markov Processes

Proposition. If the semi-Markov process is irreducible and if T_{ii} has a nonlattice distribution with finite mean, then

$$P_i \equiv \lim_{t\to\infty} P\{Z(t) = i | Z(0) = j\}$$

exists and is independent of the initial state. Furthermore,

$$P_i = \frac{\mu_i}{\mu_{ii}}.$$

Proof. Say that a cycle begins whenever the process enters state i, and say that the process is "on" when in state i and "off" when not in i. Thus we have a (delayed when $Z(0) \neq i$) alternating renewal process whose on time has distribution H_i and whose cycle time is T_{ii}. Hence, the result follows from the proposition in Chapter 3.

Semi-Markov Processes

Corollary. If the semi-Markov process is irreducible and $\mu_{ii} < \infty$, then, with probability 1,

$$\frac{\mu_i}{\mu_{ii}} = \lim_{t \to \infty} \frac{\text{amount of time in } i \text{ during } [0, t]}{t}.$$

That is, μ_i / μ_{ii} equals the long-run proportion of time in state i.

Limiting Probabilities of Semi-Markov Processes

- To compute the P_i, suppose that the embedded Markov chain $\{X_n, n \geq 0\}$ is irreducible and positive recurrent, and let its stationary probabilities be $\pi_j, j \geq 0$. That is, the $\pi_j, j \geq 0$, is the unique solution of

  $$\begin{aligned} \pi_j &= \sum_i \pi_i P_{ij}, \\ \sum_j \pi_j &= 1, \end{aligned}$$

  and π_j has the interpretation of being the proportion of the X_n's that equals j. (If the Markov chain is aperiodic, then π_j is also equal to $\lim_{n\to\infty} P\{X_n = j\}$.)

Limiting Probabilities of Semi-Markov Processes

Theorem. Suppose the semi-Markov process is irreducible and T_{ii} has a nonlattice distribution with finite mean. Suppose further that the embedded Markov chain $\{X_n, n \geq 0\}$ is positive recurrent. Then

$$P_i = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}.$$

Proof. Define the notation as follows:

- $Y_i(j)$ = amount of time spent in state i during the jth visit to that state, i, $j \geq 0$.
- $N_i(m)$ = number of visits to state i in the first m transitions of the semi-Markov process.

In terms of the above notation we see that the proportion of time in i

Limiting Probabilities of Semi-Markov Processes

during the first m transitions, call it $P_{i=m}$, is as follows:

$$\begin{aligned} P_{i=m} &= \frac{\sum_{j=1}^{N_i(m)} Y_i(j)}{\sum_i \sum_{j=1}^{N_i(m)} Y_i(j)} \\ &= \frac{\frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} \frac{Y_i(j)}{N_i(m)}}{\sum_i \frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} \frac{Y_i(j)}{N_i(m)}} \end{aligned}$$

Now since $N_i(m) \to \infty$ as $m \to \infty$, it follows from the strong law of

Limiting Probabilities of Semi-Markov Processes

large numbers that

$$\sum_{j=1}^{N_i(m)} \frac{Y_i(j)}{N_i(m)} \to \mu_i$$

and, by the strong law for renewal processes, that

$$\frac{N_i(m)}{m} \to (E[\text{number of transitions between visits to } i])^{-1} = \pi_i$$

Hence, letting $m \to \infty$ in (4.8.1) shows that

$$\lim_{m\to\infty} P_{i=m} = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}$$

and the proof is complete.

Limiting Probabilities of Semi-Markov Processes

Example.

- Consider a machine that can be in one of three states: *good condition*, *fair condition*, or *broken down*.
- Suppose that a machine in good condition will remain this way for a mean time μ_1 and will then go to either the fair condition or the broken condition with respective probabilities $\frac{3}{4}$ and $\frac{1}{4}$.
- A machine in the fair condition will remain that way for a mean time μ_2 and will then break down. A broken machine will be repaired, which takes a mean time μ_3, and when repaired will be in the good condition with probability $\frac{2}{3}$ and the fair condition with probability $\frac{1}{3}$.
- What proportion of time is the machine in each state?

Limiting Probabilities of Semi-Markov Processes

Solution. Letting the states be 1,2,3, we have that the π_i satisfy

$$\begin{aligned}
\pi_1 + \pi_2 + \pi_3 &= 1, \\
\pi_1 &= \frac{2}{3}\pi_3, \\
\pi_2 &= \frac{3}{4}\pi_1 + \frac{1}{3}\pi_3, \\
\pi_3 &= \frac{1}{4}\pi_1 + \pi_2.
\end{aligned}$$

The solution is

$$\pi_1 = \frac{4}{15}, \qquad \pi_2 = \frac{1}{3}, \qquad \pi_3 = \frac{2}{5}.$$

Hence, P_i, the proportion of time the machine is in state i, is given by

$$P_1 = \frac{4\mu_1}{4\mu_1 + 5\mu_2 + 6\mu_3},$$

Limiting Probabilities of Semi-Markov Processes

$$\begin{aligned} P_2 &= \frac{5\mu_2}{4\mu_1 + 5\mu_2 + 6\mu_3}, \\ P_3 &= \frac{6\mu_3}{4\mu_1 + 5\mu_2 + 6\mu_3}. \end{aligned}$$

Limiting Probabilities of Semi-Markov Processes

- Define
  - Y(t) = time from t until the next transition,
  - S(t) = state entered at the first transition after t.
- We are interested in computing

$$\lim_{t\to\infty} P\{Z(t)=i, Y(t)>x, S(t)=j\}.$$

- Again, we use the theory of alternating renewal processes.
- **Theorem.** If the semi-Markov process is irreducible and not lattice, then

$$\lim_{t\to\infty} P\{Z(t)=i, Y(t)>x, S(t)=j|Z(0)=k\}$$
$$= \frac{P_{ij}\int_x^\infty \overline{F}_{ij}(y)dy}{\mu_{ii}}.$$

Limiting Probabilities of Semi-Markov Processes

Proof.

- Say that a cycle begins each time the process enters state i and say that it is "on" if the state is i and it will remain i for at least the next x time units and the next state is j. Say it is "off" otherwise. Thus we have an alternating renewal process.
- Conditioning on whether the state after i is j or not, we see that

  $$E[\text{“on” time in a cycle}] = P_{ij}E[(X_{ij} - x)^+],$$

  where X_{ij} is a random variable having distribution F_{ij} and representing the time to make a transition from i to j, and $y^+ = \max(0, y)$.
- Hence

  $$E[\text{“on” time in cycle}] \quad = \quad P_{ij}\int_0^\infty P\{X_{ij} - x > a\}da$$

Limiting Probabilities of Semi-Markov Processes

$$
\begin{aligned}
&= P_{ij} \int_0^\infty \overline{F}_{ij}(a+x)da \\
&= P_{ij} \int_x^\infty \overline{F}_{ij}(y)dy.
\end{aligned}
$$

As $E[\text{cycle time}] = \mu_{ii}$, the result follows from alternating renewal processes.

Corollary. If the semi-Markov process is irreducible and not lattice, then

$$\lim_{t\to\infty} P\{Z(t)=i, Y(t)>x|Z(0)=k\} = \int_x^\infty \overline{H}_i(y)dy/\mu_{ii}.$$