Semi-Markov Processes

- A *semi-Markov process* is one that changes states in accordance with a Markov chain but takes a random amount of time between changes.

- More specifically, consider a stochastic process with states 0, 1, ..., which is such that, whenever it enters state $i, i \geq 0$:
  - The next state it will enter is state $j$ with probability $P_{ij}, i, j \geq 0$.
  - Given that the next state to be entered is state $j$, the time until the transition from $i$ to $j$ occurs has distribution $F_{ij}$.

If we let $Z(t)$ denote the state at time $t$, then $\{Z(t), t \geq 0\}$ is called a semi-Markov process.

- Thus a semi-Markov process does not possess the Markovian property that given the present state the future is independent of the past.

- In predicting the future not only would we want to know the present state, but also the length of time that has been spent in that state.
A Markov chain is a semi-Markov process in which

\[ F_{ij}(t) = \begin{cases} 
0 & t < 1 \\
1 & t \geq 1.
\end{cases} \]

That is, all transition times of a Markov chain are identically 1.

Let \( H_i \) denote the distribution of time that the semi-Markov process spends in state \( i \) before making a transition. That is, by conditioning on the next state, we see

\[ H_i(t) = \sum_j P_{ij} F_{ij}(t), \]

and let \( \mu_i \) denote its mean. That is,

\[ \mu_i = \int_0^\infty x dH_i(x). \]
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- If we let $X_n$ denote the $n$th state visited, then $\{X_n, n \geq 0\}$ is a Markov chain with transition probabilities $P_{ij}$. It is called the *embedded* Markov chain of the semi-Markov process. We say that the semi-Markov process is *irreducible* if the embedded Markov chain is irreducible as well.

- Let $T_{ii}$ denote the time between successive transitions into state $i$ and let $\mu_{ii} = E[T_{ii}]$. By using the theory of alternating renewal processes, we could derive an expression for the limiting probabilities of a semi-Markov process.
Proposition. If the semi-Markov process is irreducible and if $T_{ii}$ has a nonlattice distribution with finite mean, then

$$P_i \equiv \lim_{t \to \infty} P\{Z(t) = i | Z(0) = j\}$$

exists and is independent of the initial state. Furthermore,

$$P_i = \frac{\mu_i}{\mu_{ii}}.$$

Proof. Say that a cycle begins whenever the process enters state $i$, and say that the process is “on” when in state $i$ and “off” when not in $i$. Thus we have a (delayed when $Z(0) \neq i$) alternating renewal process whose on time has distribution $H_i$ and whose cycle time is $T_{ii}$. Hence, the result follows from the proposition in Chapter 3.
Corollary. If the semi-Markov process is irreducible and $\mu_{ii} < \infty$, then, with probability 1,

$$\frac{\mu_i}{\mu_{ii}} = \lim_{t \to \infty} \frac{\text{amount of time in } i \text{ during } [0, t]}{t}.$$  

That is, $\mu_i/\mu_{ii}$ equals the long-run proportion of time in state $i$. 
Limiting Probabilities of Semi-Markov Processes

- To compute the $P_i$, suppose that the embedded Markov chain \( \{X_n, n \geq 0\} \) is irreducible and positive recurrent, and let its stationary probabilities be $\pi_j, j \geq 0$. That is, the $\pi_j, j \geq 0$, is the unique solution of

$$
\pi_j = \sum_i \pi_i P_{ij},
$$

$$
\sum_j \pi_j = 1,
$$

and $\pi_j$ has the interpretation of being the proportion of the $X_n$’s that equals $j$. (If the Markov chain is aperiodic, then $\pi_j$ is also equal to $\lim_{n \to \infty} P\{X_n = j\}$.)

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Limiting Probabilities of Semi-Markov Processes

Theorem. Suppose the semi-Markov process is irreducible and $T_{ii}$ has a nonlattice distribution with finite mean. Suppose further that the embedded Markov chain $\{X_n, n \geq 0\}$ is positive recurrent. Then

$$P_i = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}.$$ 

Proof. Define the notation as follows:

- $Y_i(j) =$ amount of time spent in state $i$ during the $j$th visit to that state, $i, j \geq 0$.
- $N_i(m) =$ number of visits to state $i$ in the first $m$ transitions of the semi-Markov process.

In terms of the above notation we see that the proportion of time in $i$
during the first $m$ transitions, call it $P_{i=m}$, is as follows:

$$P_{i=m} = \frac{\sum_{j=1}^{N_i(m)} Y_i(j)}{\sum_{i} \sum_{j=1}^{N_i(m)} Y_i(j)}$$

$$= \frac{\frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} Y_i(j)}{\sum_{i} \frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} Y_i(j)}$$

Now since $N_i(m) \to \infty$ as $m \to \infty$, it follows from the strong law of
large numbers that
\[
\sum_{j=1}^{N_i(m)} \frac{Y_i(j)}{N_i(m)} \to \mu_i
\]

and, by the strong law for renewal processes, that
\[
\frac{N_i(m)}{m} \to (E[\text{number of transitions between visits to } i])^{-1} = \pi_i
\]

Hence, letting \( m \to \infty \) in (4.8.1) shows that
\[
\lim_{m \to \infty} P_{i=m} = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}
\]

and the proof is complete.
Limiting Probabilities of Semi-Markov Processes

Example.

- Consider a machine that can be in one of three states: *good condition*, *fair condition*, or *broken down*.

- Suppose that a machine in good condition will remain this way for a mean time $\mu_1$ and will then go to either the fair condition or the broken condition with respective probabilities $\frac{3}{4}$ and $\frac{1}{4}$.

- A machine in the fair condition will remain that way for a mean time $\mu_2$ and will then break down. A broken machine will be repaired, which takes a mean time $\mu_3$, and when repaired will be in the good condition with probability $\frac{2}{3}$ and the fair condition with probability $\frac{1}{3}$.

- What proportion of time is the machine in each state?
Limiting Probabilities of Semi-Markov Processes

Solution. Letting the states be 1,2,3, we have that the $\pi_i$ satisfy

$$\pi_1 + \pi_2 + \pi_3 = 1,$$

$$\pi_1 = \frac{2}{3} \pi_3,$$

$$\pi_2 = \frac{3}{4} \pi_1 + \frac{1}{3} \pi_3,$$

$$\pi_3 = \frac{1}{4} \pi_1 + \pi_2.$$

The solution is

$$\pi_1 = \frac{4}{15}, \quad \pi_2 = \frac{1}{3}, \quad \pi_3 = \frac{2}{5}.$$

Hence, $P_i$, the proportion of time the machine is in state $i$, is given by

$$P_1 = \frac{4\mu_1}{4\mu_1 + 5\mu_2 + 6\mu_3},$$
\[ P_2 = \frac{5\mu_2}{4\mu_1 + 5\mu_2 + 6\mu_3}, \]
\[ P_3 = \frac{6\mu_3}{4\mu_1 + 5\mu_2 + 6\mu_3}. \]
Limiting Probabilities of Semi-Markov Processes

- Define
  - $Y(t) =$ time from $t$ until the next transition,
  - $S(t) =$ state entered at the first transition after $t$.

- We are interested in computing

  $$\lim_{t \to \infty} P\{Z(t) = i, Y(t) > x, S(t) = j\}.$$ 

- Again, we use the theory of alternating renewal processes.

- **Theorem.** If the semi-Markov process is irreducible and not lattice, then

  $$\lim_{t \to \infty} P\{Z(t) = i, Y(t) > x, S(t) = j | Z(0) = k\} = \frac{P_{ij} \int_x^\infty F_{ij}(y)dy}{\mu_{ii}}.$$
Limiting Probabilities of Semi-Markov Processes

Proof.

- Say that a cycle begins each time the process enters state $i$ and say that it is “on” if the state is $i$ and it will remain $i$ for at least the next $x$ time units and the next state is $j$. Say it is “off” otherwise. Thus we have an alternating renewal process.

- Conditioning on whether the state after $i$ is $j$ or not, we see that

$$E[\text{“on” time in a cycle}] = P_{ij} E[(X_{ij} - x)^+]$$

where $X_{ij}$ is a random variable having distribution $F_{ij}$ and representing the time to make a transition from $i$ to $j$, and $y^+ = \max(0, y)$.

- Hence

$$E[\text{“on” time in cycle}] = P_{ij} \int_0^\infty P\{X_{ij} - x > a\}da$$
Limiting Probabilities of Semi-Markov Processes

\[ P_{ij} \int_0^\infty F_{ij}(a + x) \, da = P_{ij} \int_x^\infty F_{ij}(y) \, dy. \]

As \( E[\text{cycle time}] = \mu_{ii} \), the result follows from alternating renewal processes.

**Corollary.** If the semi-Markov process is irreducible and not lattice, then

\[ \lim_{t \to \infty} P\{ Z(t) = i, Y(t) > x \mid Z(0) = k \} = \int_x^\infty \frac{H_i(y)}{\mu_{ii}} \, dy. \]