Ch 8: Bivariate Distributions

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8.1 Joint distributions of two random variables

consider two r.v.s that are defined simultaneously on the same sample space.

⇒ Joint prob. mass functions ⟵

Joint prob. density functions

- Joint prob. mass functions

X, Y: two discrete r.v.s defined on the same sample space

A: the set of possible values of X ; B: the set of possible values of Y

$p(x, y) = p(X = x, Y = y) \Rightarrow$ joint prob. mass function of X and Y

① $p(x, y) \geq 0$ ② If $x \notin A$ or $y \notin B$ $p(x, y) = 0$

③ $\displaystyle\sum_{x \in A} \sum_{y \in B} p(x, y) = 1$

marginal prob. mass functions of X and Y

$\hookrightarrow$ $P_X(x) = \displaystyle\sum_{y \in B} p(x, y)$ $P_Y(y) = \displaystyle\sum_{x \in A} p(x, y)$

Ex 8-1

A small college has 90 male and 30 female professors.

An ad hoc committee of five is selected at random.

X: # of men

on this committee

Y: # of women

(a) Find joint prob. mass function of X and Y

(b) P_X, P_Y marginal

Ans:

$$p(x, y) = \frac{\binom{90}{x}\binom{30}{y}}{\binom{120}{5}}$$

if $x, y \in \{0, 1, 2, 3, 4, 5\}$

$x + y = 5$

(b) $p(x,y)=0$ if $x+y \neq 5$

$$P_X(x) = \sum_{y=0}^{5} p(x,y) = p(x, 5-x) = \frac{\binom{90}{x}\binom{30}{5-x}}{\binom{120}{5}}$$

$x, y \in \{0,1,2,3,4,5\}$

$$P_Y(y) = \sum_{x=0}^{5} p(x,y) = p(5-y, y) = \frac{\binom{90}{5-y}\binom{30}{y}}{\binom{120}{5}}$$

$$E[X] = \sum_{x \in A} x \underbrace{P_X(x)}_{\text{marginal of } X}$$

$$E[Y] = \sum_{y \in B} y \underbrace{P_Y(y)}_{\text{marginal of } Y}$$

$\Rightarrow$

Ex 8.3

$$p(x,y) = \begin{cases} \frac{1}{70} x(x+y) & \text{if } x=1,2,3, \; y=3,4 \\ 0 & \text{otherwise} \end{cases}$$

$E[X]$ $\quad$ $E[Y]$

$$P_X(x) = p(x,3) + p(x,4) = \frac{1}{35}x^2 + \frac{1}{10}x$$

$$P_Y(y) = p(1,y) + p(2,y) + p(3,y) = \frac{1}{5} + \frac{3}{35}y$$

$$E[X] = \sum_{x=1}^{3} x P_X(x) = \sum_{x=1}^{3} x\left(\frac{1}{35}x^2 + \frac{1}{10}x\right)$$

$$E[Y] = \sum_{y=3}^{4} y P_Y(y) = \sum_{y=3}^{4} y\left(\frac{1}{5} + \frac{3}{35}y\right)$$

Joint mass
marginal mass

$$E[g(X)] = \sum_{x \in A} g(x)\, p(x)$$

$$E[h(X,Y)] = \sum_{x \in A} \sum_{y \in B} h(x,y)\, p(x,y)$$

h is a function of two variables from $\mathbb{R}^2$ to $\mathbb{R}$

Corollary

$$E[X+Y] = E[X] + E[Y]$$

Proof: $h(X,Y) = x+y$

$$E[X+Y] = \sum_{x\in A}\sum_{y\in B} (x+y)\, p(x,y)$$

$$= \sum_{x\in A}\sum_{y\in B} x\,p(x,y) + \sum_{x\in A}\sum_{y\in B} y\,p(x,y)$$

$$= \underbrace{\sum_{x\in A} x\,P_X(x)} + \underbrace{\sum_{y\in B} y\,P_Y(y)}$$

discrete case

$$= E[X] + E[Y]$$

continuous case

Def: X, Y have a continuous joint distribution if there exists a nonnegative function of two variables, $f(x,y)$ on $\mathbb{R}\times\mathbb{R}$

such that for any region R in the xy-plane

$$P((X,Y) \in R) = \iint_R f(x,y)\,dx\,dy$$

$f(x,y)$: joint prob. density function

$$P(X \in A,\ Y \in B) = \int_B \int_A f(x,y)\,dx\,dy$$

$A = (-\infty, \infty) \qquad B = (-\infty, \infty)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

marginal dencity

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Joint distribution function

$$F(t,u) = P(X \le t,\ Y \le u)$$

marginal disterbution function

$$F_X(t) = P(X \le t) = P(X \le t,\ Y < \infty) \equiv F(t,\infty)$$

$$F_Y(u) = P(Y \le u) \equiv F(\infty, u)$$

joint distribution vs. joint density

$$\boxed{\frac{d^2}{dx\,dy} F(x,y) = f(x,y)}$$

$$F(x,y) = P(X \le x,\ Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(t,u)\, dt\, du$$

vs. marginal distribution function

$$F_X(x) = F(x, \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(t,u)\, dt\, du$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(t,u)\, du\, dt$$

marginal X

$$= \int_{-\infty}^{\infty} f_X(t)\, dt$$

$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)\, du$$

$$F_X'(x) = f_X(x) \qquad F_Y'(y) = f_Y(y)$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

Ex 8.4

$$f(x,y) = \begin{cases} \lambda x y^2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) λ ?

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$= \int_{y=0}^{1} \int_{x=0}^{y} \lambda x y^2 \,dx\,dy = \frac{\lambda}{3}\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{\lambda}{10}$$

$\therefore \lambda = 10$

(b) marginal prob. density functions of X and Y

$$f_X(x) = \int_x^1 f(x,y)\,dy = \int_x^1 10 x y^2\,dy = \frac{10}{3}x(1-x^3)$$

$$f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 10xy^2\,dx = 5y^4$$

(c) $E[X]$ and $E[Y]$

$$E[X] = \int_0^1 x \cdot \frac{10}{3} x(1-x^3)\,dx = \frac{5}{9}$$

$$E[Y] = \int_0^1 y \, 5y^4\,dy = \frac{5}{6}$$

Ex 8.5

$$F(x,y) = \begin{cases} 1 - \lambda e^{-\lambda(x+y)} & \text{if } x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

$\lambda > 0$

Determine if F is the joint prob. distribution function of two r.v.s X and Y.

Ans:

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y) \geq 0$$

$$\parallel$$

$$-\lambda^3 e^{-\lambda(x+y)} \quad \text{if } x>0,\ y>0$$

Since $\frac{\partial^2}{\partial x \partial y} F(x,y) < 0$

$\therefore$ F is not a joint distribution function

Def: S: a subset of the xy-plane with area $A(S)$

A point is said to be <u>randomly selected</u> from S if

for any subset R of S with area $A(R)$

the prob. that R contains the point is $\dfrac{A(R)}{A(S)}$

$\Rightarrow$ geometric probability

Ex 8.7

A man invites his fiancee to a fine hotel for a Sunday brunch.

They decide to meet in the lobby of the hotel between 11:30 AM and 12 noon

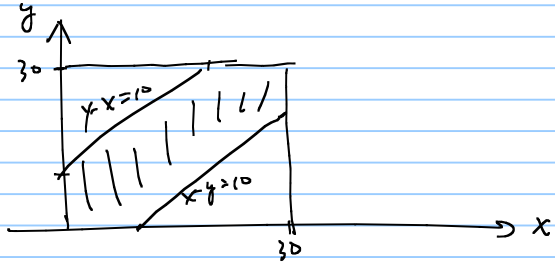
If they arrive at random times during this period,

what is the prob. that they will meet within 10 minutes ?

Ans: X and Y be the minutes past 11:30 AM

$S = \{(x,y) : 0 \le x \le 30 , 0 \le y \le 30\}$

$R = \{(x,y) \in S : |x-y| \le 10\}$



$$P(|X-Y| \le 10) = \frac{A(R)}{A(S)} = \frac{30\times30-2(\frac{1}{2}\cdot20\cdot20)}{30\times30} = \frac{5}{9}$$

Thm 8-2

$$E[h(x,y)] = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} h(x,y)f(x,y)\,dx\,dy$$

Corollary

$$E[X+Y] = E[X] + E[Y]$$

Ex 8.9

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X^2+Y^2] = \int_{y=0}^{1}\int_{x=0}^{1} (x^2+y^2) f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 \frac{3}{2}(x^2+y^2)^2\,dx\,dy = \frac{3}{2}\int_0^1\int_0^1 x^4+2x^2y^2+y^4\,dx\,dy$$

$$= \frac{14}{15}$$

8.2 Independent random variables

Two random variables X and Y are called <u>independent</u>

if $\quad p(\underbrace{X \in A}_{\text{event } E_1}, \underbrace{Y \in B}_{\text{event } E_2}) = p(X \in A)\, p(Y \in B)$

$\hookrightarrow \quad p(X \le a, Y \le b) = p(X \le a)\, p(Y \le b)$

$\hookrightarrow$

Thm 8.3 $\quad F(t, u) = F_X(t)\, F_Y(u)$

joint distribution of X and Y marginal of X marginal of Y

Discrete case

Continuous case

- Discrete case

independent

$$p(x, y) = P_X(x)\, P_Y(y) \Rightarrow (8.11)$$

joint mass   marginal mass of X marginal mass of Y

$$p(x, y) = \Pr[X = x, Y = y] = \Pr[X \in \{x\}, Y \in \{y\}]$$
$$= \Pr[X \in \{x\}] \Pr[Y \in \{y\}]$$

(8.11) $= p_X(x)\, p_Y(y)$

implies

$$p(X=x \mid Y=y) = p(X=x)$$

$$\frac{p(x,y) = p_X(x)\, p_Y(y)}{p_Y(y)} = p_X(x)$$

$$p(Y=y \mid X=x) = p(Y=y)$$

Ex 8-10

4% of the bicycle fenders produced by a stamping machine from the strips of steel need smoothing.

What is the prob. that, of the next 13 bicycle fenders stamped by this machine, two need smoothing and, of the next 20, three need smoothing?

Ans: X be the # of bicycle fenders among the first 13 that need smoothing

Y -- -- among the next 7 -- -- .

$$p(X=2, Y=1) = p(X=2)\,p(Y=1) = \binom{13}{2}(0.04)^2(0.96)^{11} \cdot \binom{7}{1}(0.04)^1(0.96)^6$$

$X \sim \text{binomial}(13, 0.04)$ ≈ 0.0175

$Y \sim \text{binomial}(7, 0.04)$

- Continuous case

X and Y are independent if and only if

$$f(x, y) = f_X(x)\, f_Y(y)$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{F_X(x)\, F_Y(y)}{\partial x \partial y}$$

Ex 8.12

Stores A and B

prob. density function of the weekly profit of each store

$$f(x) = \begin{cases} \frac{x}{4} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

the profit of one store is independent of the other

the prob. that next week one store makes at least \$500 more than the other store?

Ans: X: the profit of A

Y: the profit of B

$$p(X > Y + \tfrac{1}{2}) + p(Y > X + \tfrac{1}{2}) = 2p(X > Y + \tfrac{1}{2})$$

symmetric

$$\Downarrow$$

$f(x, y)$

$$f_X(x) = \begin{cases} \frac{x}{4} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y}{4} & 1 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x, y) = f_X(x) f_Y(y) = \begin{cases} \frac{xy}{16} & 1 < x, y < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$2P\left(X > Y + \frac{1}{2}\right) = 2P\left[(X,Y) \in \left\{(x,y) : \left| \frac{3}{2} < x < 3,\ 1 < y < x - \frac{1}{2}\right.\right\}\right]$$

$$= 2\int_{\frac{3}{2}}^{3}\int_{1}^{x-\frac{1}{2}} f(x,y)\,dy\,dx$$

$$= 2\left[\int_{\frac{3}{2}}^{3}\int_{1}^{x-\frac{1}{2}} \frac{xy}{16}\,dy\,dx\right]$$

$$= \frac{1}{8}\int_{\frac{3}{2}}^{3}\left[\frac{xy^2}{2}\right]_{1}^{x-\frac{1}{2}} dx$$

$$= \frac{1}{16}\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{3}{8}x^2\right]_{\frac{3}{2}}^{3} = \frac{549}{1024} \approx 0.54$$

Ex: 8.15

$$f(x,y) = \begin{cases} 8xy & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ X, Y independent ?

$$\underbrace{f(x,y)}_{\text{joint density}} = \underbrace{f_X(x)\, f_Y(y)}_{\text{marginal density}}$$

$$f_X(x) = \int_x^1 8xy\,dy = 4x(1-x^2)$$

$$f_Y(y) = \int_0^y 8xy\,dx = 4y^3$$

$\because f(x,y) \neq f_X(x) f_Y(y)$

$\Rightarrow$ X and Y are dependent

8.3 Conditional distributions

Discrete case

Continuous case

If the value of Y is known, the conditional prob. mass function of X

given that $Y=y$ $\quad P_{X|Y}(x|y)$

$$P_{X|Y}(x|y) = \frac{p(x,y) \rightarrow \text{joint mass}}{P_Y(y) \rightarrow \text{marginal mass of } Y}$$

$$\text{"}\sum_{x \in A} P_{X|Y}(x|y)\text{"} = \sum_{x \in A} \frac{p(x,y)}{P_Y(y)} = \frac{1}{P_Y(y)} \underbrace{\sum_{x \in A} p(x,y)}_{\text{marginal mass of } Y}$$

prob. mass function

with the set of possible values A

$$= \frac{1}{P_Y(y)} \cdot P_Y(y) = 1$$

If X and Y are independent

$$P_{X|Y}(x|y) = \frac{p(x,y)}{P_Y(y)} = \frac{P(X=x, Y=y)}{P_Y(y)} = \frac{P(X=x)P(Y=y)}{P_Y(y)} = P(X=x)$$

$$= P_X(x)$$

$F_{X|Y}(x|y)$

(conditional distribution function of X, given that $Y=y$)

$$= P(X \le x \mid Y=y)$$

$$= \sum_{t \le x} P(X=t \mid Y=y) = \sum_{t \le x} P_{X|Y}(t|y)$$

Ex 8.16

$$p(x,y) = \begin{cases} \frac{1}{15}(x+y) & x=0,1,2, \quad y=1,2 \end{cases}$$

$P_{X|Y}(x|y)$ and $p(X=0 \mid Y=2)$

① $P_{X|Y}(x|y) = \dfrac{p(x,y)}{P_Y(y)}$

$$P_Y(y) = \sum_{x=0}^{2} p(x,y) = \sum_{x=0}^{2} \left[\frac{1}{15}(x+y)\right] = \frac{1+y}{5}$$

$$\therefore P_{X|Y}(x|y) = \frac{(x+y)/15}{\frac{1+y}{5}} = \frac{x+y}{3(1+y)} \qquad x = 0, 1, 2, \text{ when } y = 1 \text{ or } 2$$

$$p(X=0 \mid Y=2) = \frac{0+2}{3(1+2)} = \frac{2}{9} \quad \#$$

conditional expectation

$$E[X \mid Y=y] = \sum_{x \in A} x \cdot P_{X|Y}(x|y)$$

$$E[h(X) \mid Y=y] = \sum_{x \in A} h(x) P_{X|Y}(x|y)$$

Ex 8.18

Calculate the expected number of aces in a randomly selected poker hand that is found to have exactly two jacks.

Ans: X : # of aces

Y : # of jacks

$$E[X \mid Y=2] = \sum_{x=0}^{3} x \cdot P_{X|Y}(x|2) = \sum_{x=0}^{3} x \frac{P(x,2)}{P_Y(2)} = \sum_{x=0}^{3} x \frac{\dfrac{\binom{4}{x}\binom{4}{2}\binom{44}{5-2-x}}{\binom{52}{5}}}{\dfrac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}}$$

$$= \sum_{x=0}^{3} x \frac{\binom{4}{x}\binom{44}{3-x}}{\binom{48}{3}} \approx 0.25$$

Continuous case

$f_{X|Y}(x|y)$

(conditional prob. density function of X given that $Y = y$)

$$= \frac{f(x,y) \rightarrow \text{joint density}}{f_Y(y) \rightarrow \text{marginal density of } Y}$$

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_Y(y)}\,dx = \frac{1}{f_Y(y)} \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dx}_{\text{marginal density of } Y}$$

$$= \frac{1}{f_Y(y)} \cdot f_Y(y) = 1$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)\,f_Y(y)}{f_Y(y)} = f_X(x)$$

if X and Y are independent

conditional prob. distribution function

$$F_{X|Y}(x|y) = P[X \le x \mid Y = y] = \int_{-\infty}^{x} f_{X|Y}(t|y)\,dt$$

$$\frac{d}{dx} F_{X|Y}(x|y) = f_{X|Y}(x|y)$$

Ex 8.21

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & 0<x<1 \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{2}(x^2+y^2)}{\int_0^1 f(x,y)\,dx} = \frac{\frac{3}{2}(x^2+y^2)}{\int_0^1 \frac{3}{2}(x^2+y^2)\,dx}$$

$$= \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}y^2+\frac{1}{2}} = \frac{3(x^2+y^2)}{3y^2+1}$$

Ex 8.22

First, a point Y is selected at random from $(0,1)$.

Then another point X is chosen at random from $(0, Y)$

Find the prob. density function of X ?

$$f(x,y) \Rightarrow f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

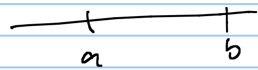
$f(x,y)$?

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \qquad \therefore f(x,y) = f_Y(y)\, f_{X|Y}(x|y)$$

$$f_Y(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & 0<y<1,\ 0<x<y \\ 0 & \text{elsewhere} \end{cases}$$

$$f(x) = \frac{1}{b-a}$$



uniform

$$f(x,y) = 1 \cdot \frac{1}{y}$$

$$\therefore f_X(x) = \int_x^1 \frac{1}{y}\,dy = \ln y \Big|_x^1 = \ln 1 - \ln x$$

$$= 0 - \ln x = -\ln x$$

$$\therefore f_X(x) = \begin{cases} -\ln x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Ex 8.23

$$f_{X,Y}(x,y) = \frac{x+y}{1+y} e^{-x}$$

$$P(X<1 \mid Y=2)$$

$$f_{X|Y}(x|2) = \frac{x+2}{3} e^{-x}$$

$$\int_0^1 f_{X|Y}(x|2)\,dx = \frac{1}{3}\left[\int_0^1 x e^{-x}\,dx + \int_0^1 2e^{-x}\,dx\right]$$

$$= 1 - \frac{4}{3}e^{-1} \approx 0.509$$

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

(conditional expectation of X

given $Y=y$)

$$E[h(X) \mid Y=y] = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y)\,dx$$

conditional variance of X given that Y = y

$$Var[X] = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (X - E[X])^2 f_X(x)\, dx$$

σ_X^2

$$\sigma^2_{X|Y=y} = \int_{-\infty}^{\infty} (X - E[X|Y=y])^2 f_{X|Y}(x|y)\, dx$$

Ex 8.24 $f(x,y) = \begin{cases} e^{-y} & \text{if } y>0,\ 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$

$E[X \mid Y=2]$?

Ans:
$$E[X \mid Y=2] = \int_0^1 x f_{X|Y}(x|2)\,dx$$

$$= \int_0^1 x \frac{f(x,2)}{f_Y(2)}\,dx = \int_0^1 x \frac{e^{-2}}{f_Y(2)}\,dx$$

$$f_Y(2) = \int_0^1 f(x,2)\,dx = \int_0^1 e^{-2}\,dx = e^{-2}$$

$$\therefore E[X \mid Y=2] = \int_0^1 x \frac{e^{-2}}{e^{-2}}\,dx = \frac{1}{2}$$

ex 8.25

The lifetimes of batteries are identically distributed with F and f.

$\Rightarrow$ expected lifetime of an s-hour-old battery ?

in terms of F, f, s

Ans: X: the lifetime of the s-hour-old battery

$E[X \mid X > s]$?

$$\begin{cases} F_{X|X>s}(t) = P(X \le t \mid X > s] \\ f_{X|X>s}(t) \end{cases}$$

$$E[X \mid X>s] = \int_0^\infty t \underbrace{f_{X \mid X>s}(t)} dt$$

$f_{X \mid X>s}(t)$?

$$\bar{F}_{X \mid X>s}(t) = P(X \le t \mid X>s) = \frac{P(X \le t, X>s)}{P(X>s)} = \begin{cases} 0 & t \le s \\ \dfrac{P(s<X \le t)}{P(X>s)} & t>s \end{cases}$$

$$= \begin{cases} 0 & t \le s \\ \dfrac{\bar{F}(t)-\bar{F}(s)}{1-\bar{F}(s)} & t>s \end{cases}$$

$$\therefore f_{X \mid X>s}(t) = \frac{d\bar{F}_{X \mid X>s}(t)}{dt}$$

$$= \begin{cases} 0 & t \le s \\ \dfrac{f(t)}{1-F(s)} & t > s \end{cases}$$

$$E[X \mid X>s] = \int_0^{\infty} t\, f_{X\mid X>s}(t)\, dt = \frac{1}{1-F(s)} \int_s^{\infty} t f(t)\, dt \quad \#$$