Ch7: Special Continuous Distributions

7.1 Uniform random variables

- Suppose that X is the value of the random point selected from an interval (a, b).
- Then X is called a **random variable** over (a, b).
- Let F and f be probability distribution and density functions of X, respectively. Clearly,

7.1 Uniform random variables

$$F(t)=\begin{cases}0 & t<a\\ & a\le t<b\\ 1 & t\ge b\end{cases}$$

$$f(t)=F^{'}(t)=\begin{cases} & \text{if } a<t<b\\ 0 & \text{otherwise}\end{cases}$$

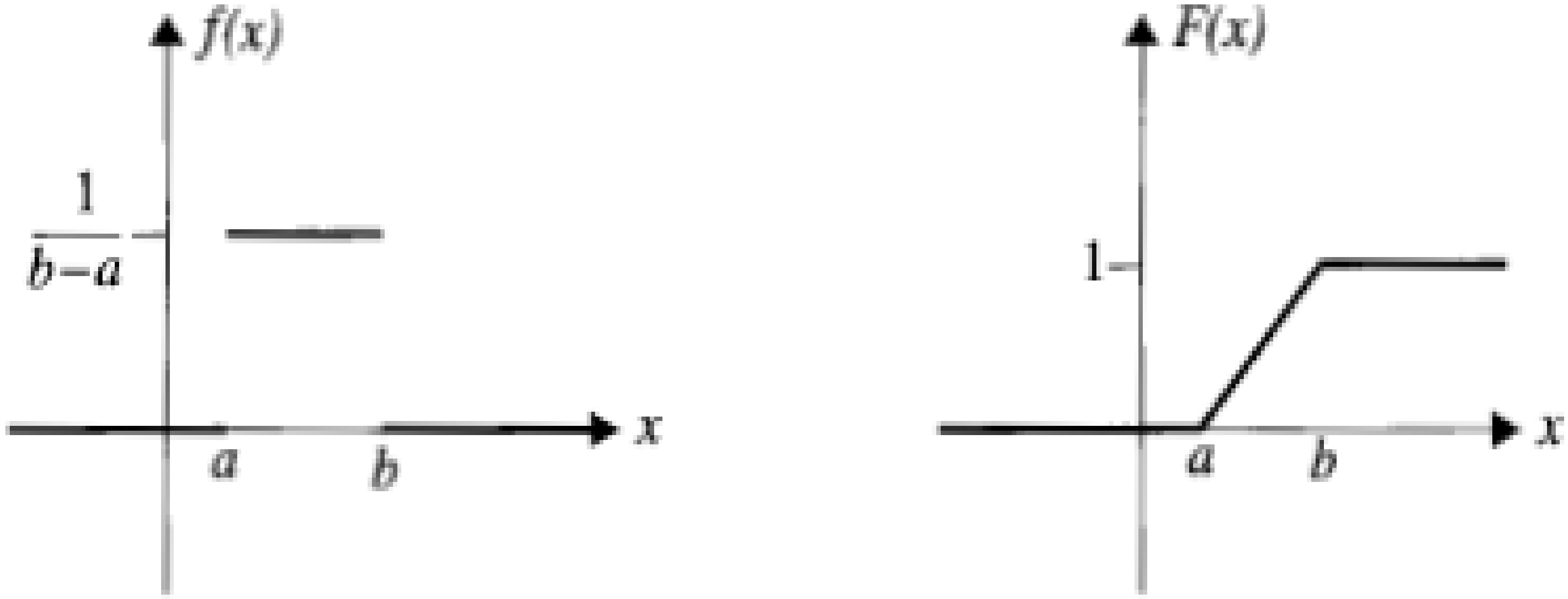


Figure 7.1 Density and distribution functions of a uniform random variable.

- If *X* is uniformly distributed over an interval *(a, b)*, then

$$E(X) =$$

$$Var(X) =$$

$$\sigma_X = \frac{b-a}{\sqrt{12}}$$

Example 7.1

- Starting at 5:00 A.M., every half hour there is a flight from San Francisco airport to Los Angeles International airport.
- Suppose that none of these planes is completely sold out and that they always have room for passengers.
- A person who wants to fly to L.A. arrives at the airport at a random time between 8:45 A.M. and 9:45 A.M.
- Find the probability that she waits (a) at most 10 minutes; (b) at least 15 minutes.

Example 7.1

1. Let the passenger arrive at the airport X minutes pass 8:45. Then X is a uniform random variable over the interval (0, 60). Hence the density function of X is given by

$$f(x) = \begin{cases} & \\ 0 & \text{elsewhere} \end{cases}$$

2. The passenger waits at most 10 minutes :

3. The passenger waits at least 15 minutes :

$$= \int_{15}^{30} \frac{1}{60} dx + \int_{45}^{60} \frac{1}{60} dx = \frac{1}{2}$$

7.2 Normal random variables

- **Theorem 7.1 (De Moivre-Laplace Theorem)** *Let X be a binomial random variable with parameters n and p. Then for any numbers a and b, a < b,*

$$\lim_{n\to\infty} P\left(a < \frac{X - np}{\sqrt{np(1-p)}} < b\right) =$$

- Note that *np* and $\sqrt{np(1-p)}$ appearing in this formula are, respectively, *E(X)* and σ_X.

- By this theorem, if *X* is a binomial random variable with parameters *(n, p)*, the sequence of probabilities

$$P\left(\frac{X-np}{\sqrt{np(1-p)}}\le t\right), \qquad n=1,2,3,4\ldots,$$

converges to $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-x^2/2}dx$ *,* where the function is a distribution function itself.

Definition

- *A random variable X is called* ***standard normal*** *if its distribution function is Φ, that is, if*

$$P(X \le t) = \Phi(t) \equiv$$

- By the fundamental theorem of calculus, f, the density function of a standard normal random variable, is given by

$$f(x) =$$

- The standard normal density function is a bell-shaped curve that is symmetric about the (see Figure 7.5).

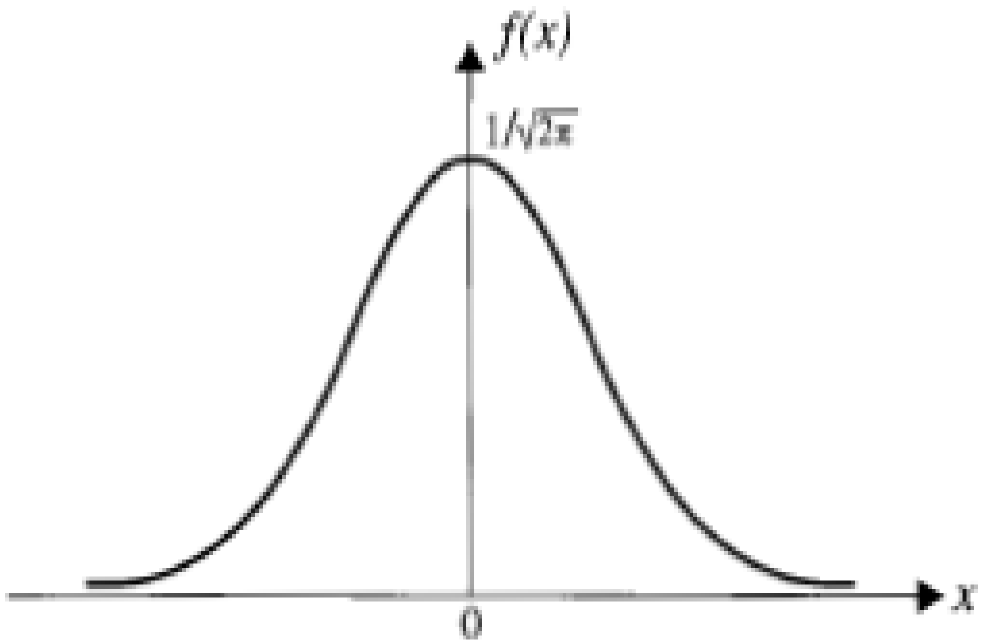


Figure 7.5 Graph of the standard normal density function.

- Since Φ is the distribution function of the standard normal random variable, $\Phi(t)$ is the area under this curve from $-\infty$ to t.
- Because $\Phi(\infty) =$ and the curve is symmetric about the y-axis, $\Phi(0) =$ Moreover,

  $\boldsymbol{\Phi(-t) =}$

Correction for Continuity

- The De Moivre-Laplace theorem approximates the distribution of a discrete random variable by that of a continuous one.
- Let X be a discrete random variable with probability mass function $p(x)$, and suppose that we want to find $P(i \leq X \leq j)$, $i < j$.
- Consider the **histogram** of X, as sketched in Figure 7.6 from i to j. In that figure the base of each rectangle equals 1, and the height (and therefore the area) of the rectangle with the base midpoint k is

- Thus the sum of the areas of all rectangles is , which is the exact value of $P(i \leq X \leq j)$.

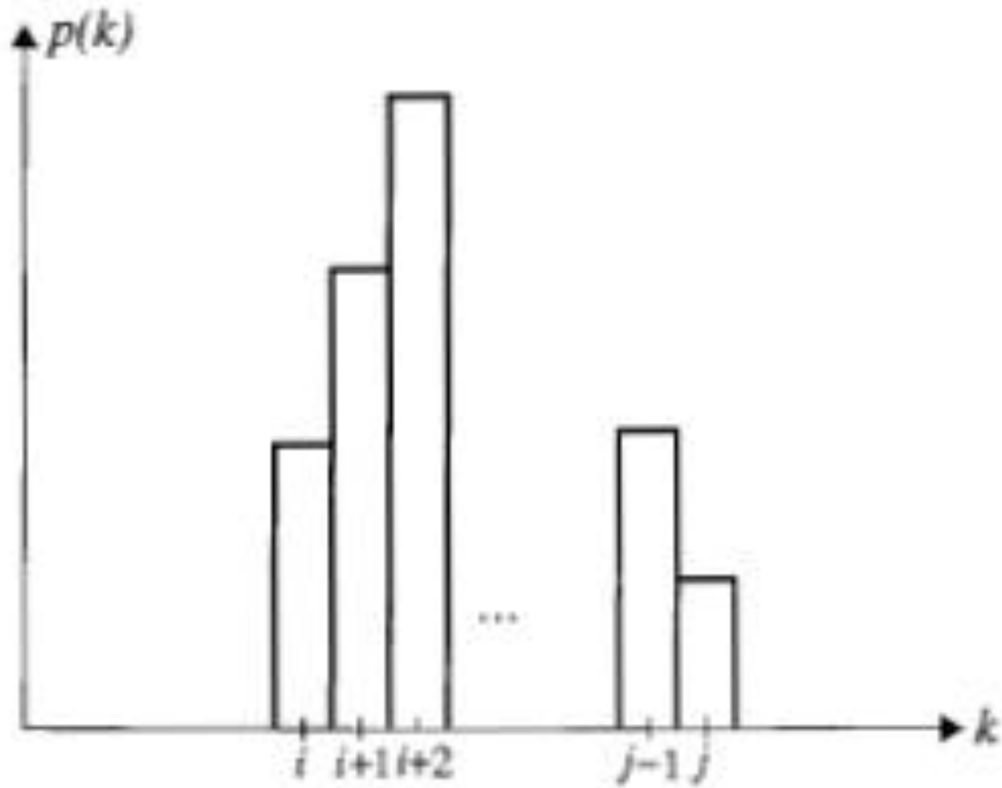


Figure 7.6 Histogram of X from i to j.

- Now suppose that $f(x)$, the density function of a continuous random variable, sketched in Figure 7.7, is a good approximation to $p(x)$.
- Then, as this figure shows, $P(i \leq X \leq j)$, the sum of the areas of all rectangles of the figure, is approximately the area under $f(x)$ from $i - 1/2$ to $j + 1/2$ rather than from i to j. That is,

$$P(i \leq X \leq j) \approx$$

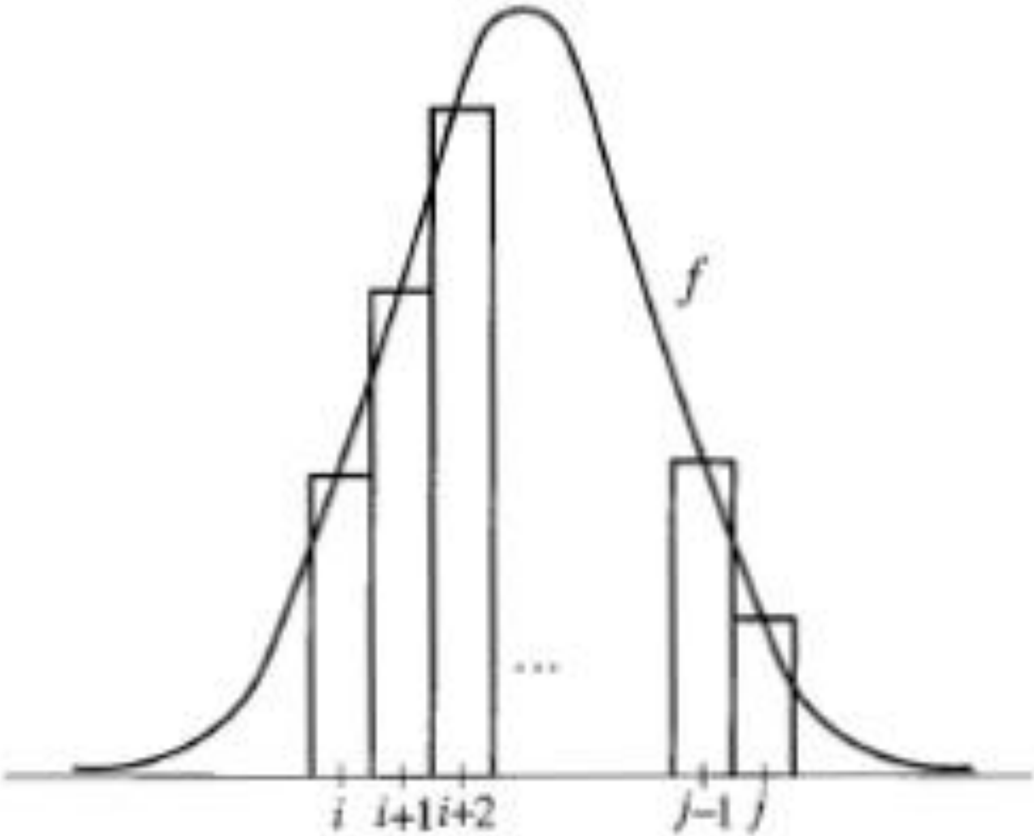


Figure 7.7 Histogram of X and the density function f.

- This adjustment is called
and is necessary for approximation of the distribution of a discrete random variable with that of a continuous one.
- Similarly, the following corrections for continuity are made to calculate the given probabilities.

$$P(X = k) \approx$$

$$P(X \geq i) \approx$$

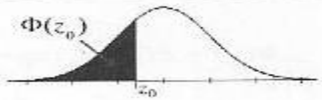
$$P(X \leq j) \approx$$

- In real-world problems, or even sometimes in theoretical ones, to apply the De Moivre-Laplace theorem, we need to calculate the numerical values of for some real numbers *a* and *b*.
- Since has no antiderivative in terms of elementary functions, such integrals are approximated by numerical techniques.
- Table 1 and Table 2 of the Appendix

Tables 1 and 2

Table 1 Area under the Standard Normal Distribution to the Left of z_0: Negative z_0

$$\Phi(z_0) = P(Z \le z_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_0} e^{-x^2/2}\, dx$$



Note that for $z_0 \le -3.90$, $\Phi(z_0) = P(Z \le z_0) \approx 0$.

z_0	0	1	2	3	4	5	6	7	8	9
−3.8	.0001	0001	0001	0001	0001	0001	0001	0001	0001	0001
−3.7	.0001	0001	0001	0001	0001	0001	0001	0001	0001	0001
−3.6	.0002	0002	0001	0001	0001	0001	0001	0001	0001	0001
−3.5	0002	0002	0002	0002	0002	0002	0002	0002	0002	0002
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table 2 Area under the Standard Normal Distribution to the Left of z_0: Positive

$$\Phi(z_0) = P(Z \le z_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_0} e^{-x^2/2}\, dx$$

z_0	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9889	.9889	.9889	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999

Note that for $z_0 > 3.89$, $\Phi(z_0) = P(Z \le z_0) \approx 1$.

Example 7.4

- Suppose that of all the clouds that are seeded with silver iodide, 58% show splendid growth.
- If 60 clouds are seeded with silver iodide, what is the probability that exactly 35 show splendid growth?

1. Let X be the number of clouds that show splendid growth. Then *E(X)* =
    = 34.80 and $\sigma_X = \qquad = 3.82$

2. By correction for continuity and De Moivre-Laplace theorem,

$$P(X = 35) \approx$$

$$= P(\frac{34.5 - 34.8}{3.82} < \frac{X - 34.8}{3.82} < \frac{35.5 - 34.8}{3.82})$$

$$=$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.18} e^{-x^2/2} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.08} e^{-x^2/2} dx$$

$$= \Phi(0.18) - \Phi(-0.08) = 0.5714 - 0.4681 = 0.1033$$

3. The exact value of *P(X = 35) is*

$$\approx 0.1039$$

The answer obtained by the De Moivre-Laplace approximation is very close to the actual probability.

- $E(X) = 0$
- $\mathrm{Var}(X) = 1$
- $\sigma_X = 1$

Expected Value of Normal

- Let X be a standard normal random variable. Then

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} dx = 0$$

- because the integrand, , is a finite odd function, and the integral is taken from -∞ to +∞.

Variance of Normal

- To calculate Var(X), note that

  $$E(X^2) =$$

- Using integration by parts, we get
  (let u = x, dv = $xe^{-x^2/2}dx$)

  $$\int_{-\infty}^{\infty} xxe^{-x^2/2}dx = \left[-xe^{-x^2/2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}dx = 0 + \sqrt{2\pi} = \sqrt{2\pi}$$

- Therefore, $E(X^2)=1$, Var(X)= , and

  $$\sigma_x = \sqrt{Var(X)} = 1$$

- We have shown that the expected value of a standard normal random variable is 0. Its standard deviation is 1.

Normal random variables

- When it comes to the analysis of data, due to the lack of parameters in the *standard normal distribution*, it cannot be used. To overcome this difficulty, mathematicians generalized the standard normal distribution by introducing the following density function.
- **Definition** *A random variable X is called , with parameters μ and σ, if its density function is given by*

$$f(x) =$$

Gaussian distribution

- If X is a normal random variable with parameters μ and σ, we write
- One of the first applications of $N(\mu, \sigma^2)$ was given by Gauss in 1809. Gauss used $N(\mu, \sigma^2)$ to model the errors of observations in astronomy.
- For this reason, the normal distribution is sometimes called **the distribution.**

Lemma 7.1

- If X ~ N(μ,σ²), then Z = (X-μ)/σ is N(0,1). That is, if X ~ N(μ,σ²), the standardized X is N(0,1).
- Proof:

  We show that the distribution function of Z is
  $$(1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-y^2/2}dy$$
- Note that $P(Z \le x) =$
  $$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\sigma x+\mu} \exp\left[\frac{-(t-\mu)^2}{2\sigma^2}\right]dt$$

Lemma 7.1

- Let y ; then dt=σdy and we get

$$P(Z \le x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\sigma dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2} dy$$

- The parameters μ and σ that appear in the formula of the density function of X are its expected value and standard deviation, respectively.
- Note that is $N(0, 1)$ and $X=$. Hence
  - $E[X]= \quad =\sigma E[Z]+\mu = \mu$
  - $Var[X]=Var[\sigma Z+\mu]= \quad =\sigma^2$

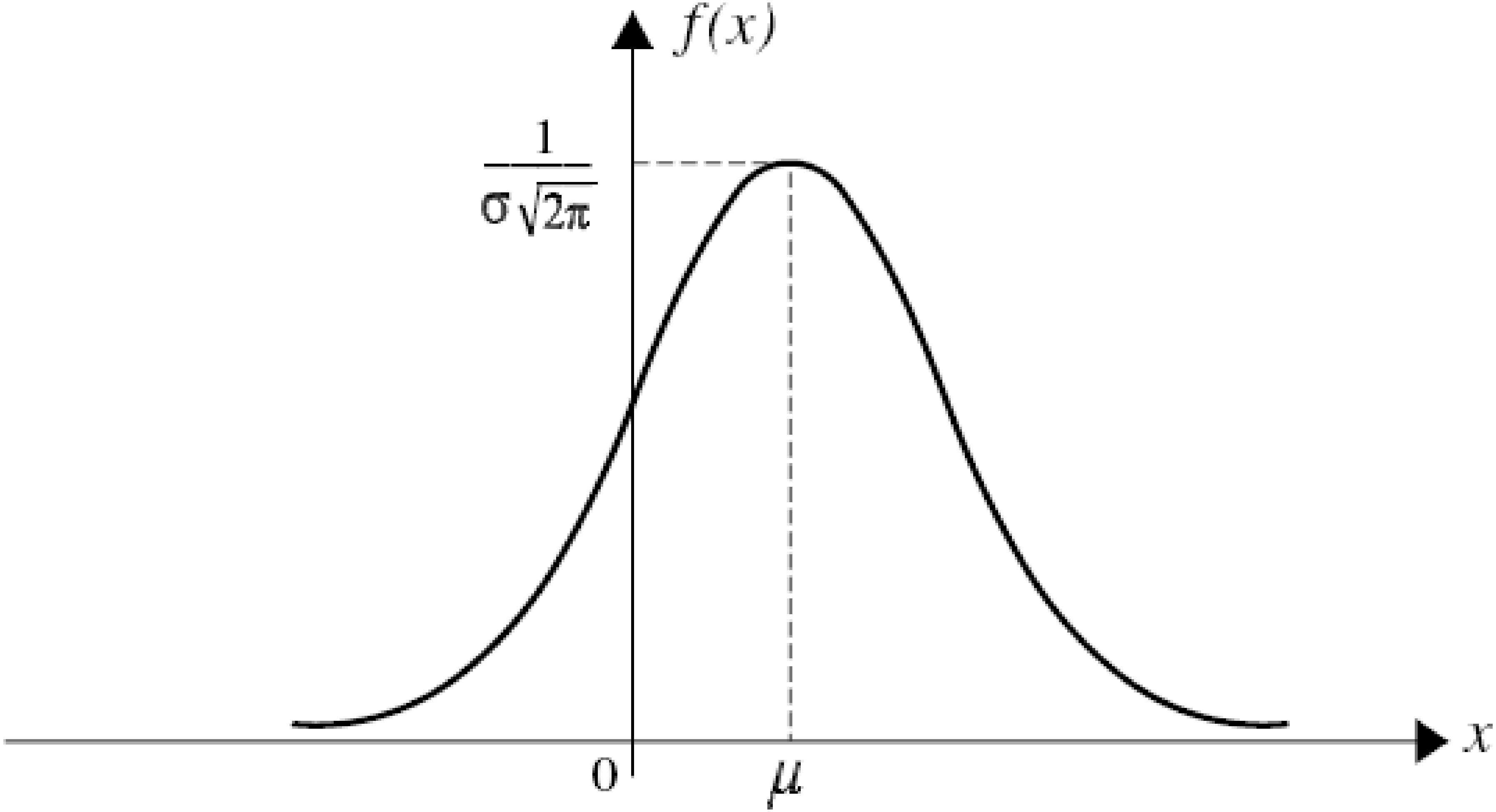


Figure 7.8 Density of $N(\mu, \sigma^2)$.

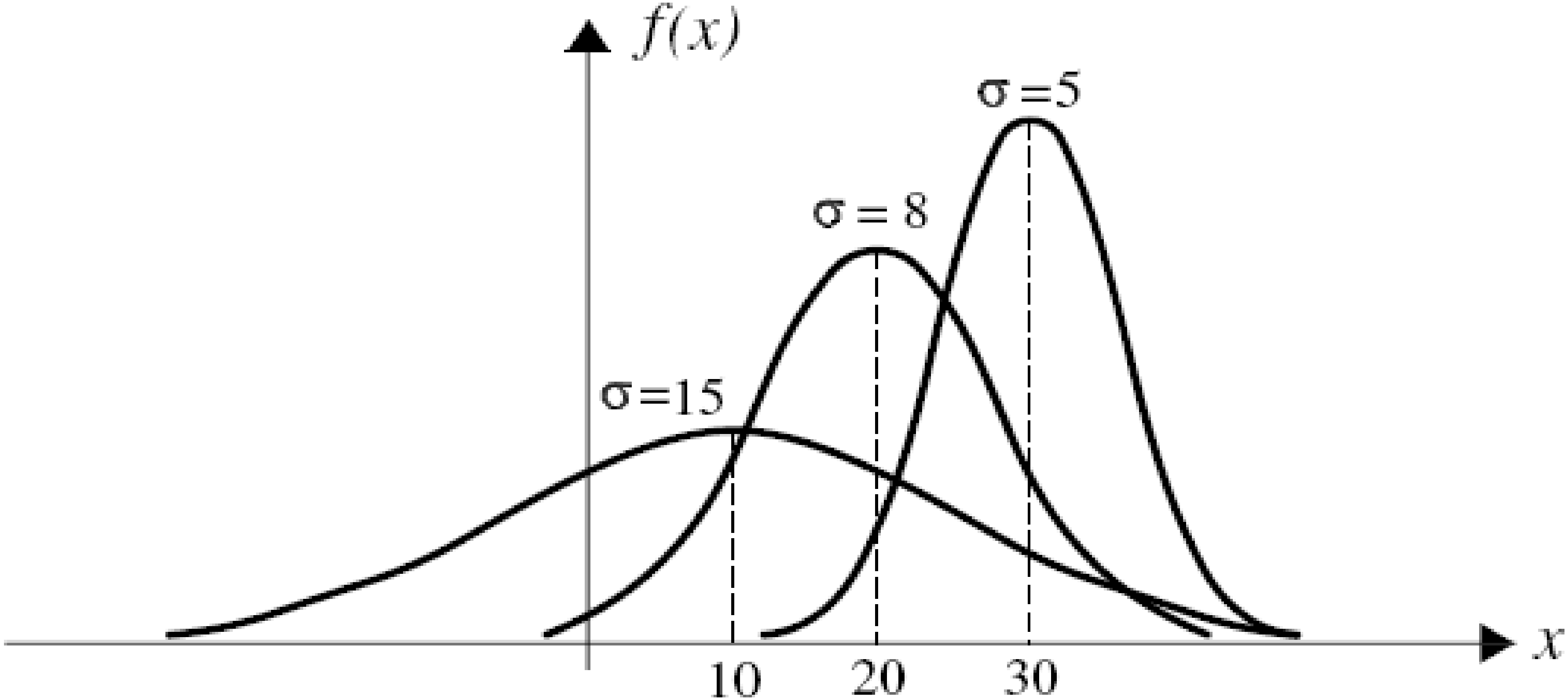


Figure 7.9 Different normal densities with specified parameters.

- The transformation
  enables us to use Tables 1 and 2 of the Appendix to calculate the probabilities concerning X.

Example 7.5

- Suppose that a Scottish soldier’s chest size is normally distributed with mean 39.8 and standard deviation 2.05 inches, respectively.
- What is the probability that of 20 randomly selected Scottish soldiers, five have a chest of at least 40 inches?

1. Let p be the probability that a randomly selected Scottish soldier has a chest of 40 or more inches. If X is the normal random variable with mean 39.8 and standard deviation 2.05

2. $$p = \qquad = P\left(\frac{X-39.8}{2.05} \geq \frac{40-39.8}{2.05}\right) = P\left(\frac{X-39.8}{2.05} \geq 0.1\right)$$

$$= \qquad = 1-\Phi(0.1) \approx 1-0.5398 \approx 0.46$$

3. Therefore, the probability that of 20 randomly selected Scottish soldiers, five have a chest of at least 40 inches is

$$\approx 0.33$$

Example 7.7

- The scores on an achievement test given to 100,000 students are normally distributed with mean 500 and standard deviation 100.
- What should the score of a student be to place him among the top 10% of all students?

1. Letting X be a normal random variable with mean 500 and standard deviation 100, we must find x so that $P(X \geq x) = 0.10$ or $P(X < x) = 0.90$.

2.

$$= 0.90 \Rightarrow \qquad = 0.90$$

3. From Table 2 of the Appendix, we have that $\Phi(1.28) \approx 0.8997$, implying that $(x - 500)/100 \approx 1.28$. This gives $x \approx 628$; therefore, a student should earn 628 or more to be among the top 10% of the students.

7.3 Exponential random variables

- A typical use of the distribution arises in situations where “event” occur at certain points in time.
  - E.g., an event is the occurrence of an earthquake
- The distribution is often used to model the number of events occurring in any interval of length *t*.

- Let us call the interval $[0,t]$ and denote by the number of events occurring in that interval.
  - To obtain an expression for $P\{N(t)=k\}$, we start by breaking the interval $[0,t]$ into n nonoverlapping subintervals each of length
- Let the event occurrence in the n subintervals be independent, and assume that in any given subinterval
  - one event occurs with probability p
  - no events occur with probability

- Then *N*(*t*) has the binomial distribution with mean
- Let $n \to \infty, p \to 0, np = \lambda t$. We know that *N*(*t*) is approximately a Poisson random variable with mean and probability mass function

$$P(N(t) = i) = \qquad\qquad , \quad i = 0, 1, 2, 3, \cdots$$

- In the Poisson distribution, is interpreted as *the occurrence rate of events* or the *expected number of events per unit time*.

Exponential r.v. VS. Poisson r.v.

- The set of Poisson random variables $\{N(t): t \geq 0\}$ form a
- Let X_1 be the time of the first event, X_2 be the elapsed time between the first and the second events, X_3 be the elapsed time between the second and third events, and so on.
- The sequence of random variables $\{X_1, X_2, X_3, \ldots\}$ is called the **sequence of**
  of the Poisson process $\{N(t): t \geq 0\}$.

$$P(N(t)=n)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

- For $t \geq 0$,

  $P(X_1 > t) = \qquad\qquad = e^{-\lambda t}$

- Therefore,

  $P(X_1 \leq t) = \qquad\qquad = 1 - e^{-\lambda t}$

- Because of the independent and identical Bernoulli trials, we can expect that the interarrival time of any two consecutive events has the same distribution as X_1; that is, the sequence $\{X_1, X_2, X_3, \ldots\}$ is identically distributed.
- Therefore, for all $n \geq 1$,

$$P(X_n \leq t) = P(X_1 \leq t) = \begin{cases} & \mathrm{t} \geq 0 \\ 0 & \mathrm{t} < 0 \end{cases}$$

- Let

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & \mathrm{t} \geq 0 \\ 0 & \mathrm{t} < 0 \end{cases}$$

for some $\lambda > 0$. Then F is the distribution function of X_n for all $n \geq 1$. It is called **distribution.**

$$f(t) = F'(t) = \begin{cases} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- And

$$\int_0^\infty \lambda e^{-\lambda t} dt = \qquad = \lim_{b\to\infty} \left[-e^{-\lambda t} \right]_0^b$$

$$= \lim_{b\to\infty} (1 - e^{-\lambda b}) = 1$$

f is a density function.

Definition

- *A continuous random variable X is called* ***exponential*** *with parameter λ > 0 if its density function is given by*

$$f(t) = \begin{cases} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- For an exponential random variable with parameter λ,

$$E(X) = \sigma_X =$$

$$Var(X) =$$

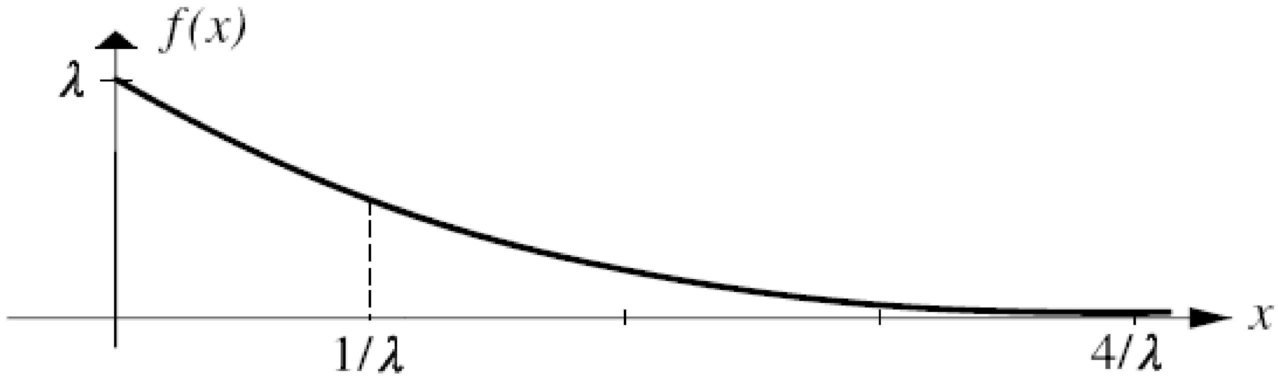


Figure 7.10 Exponential density function with parameter λ.

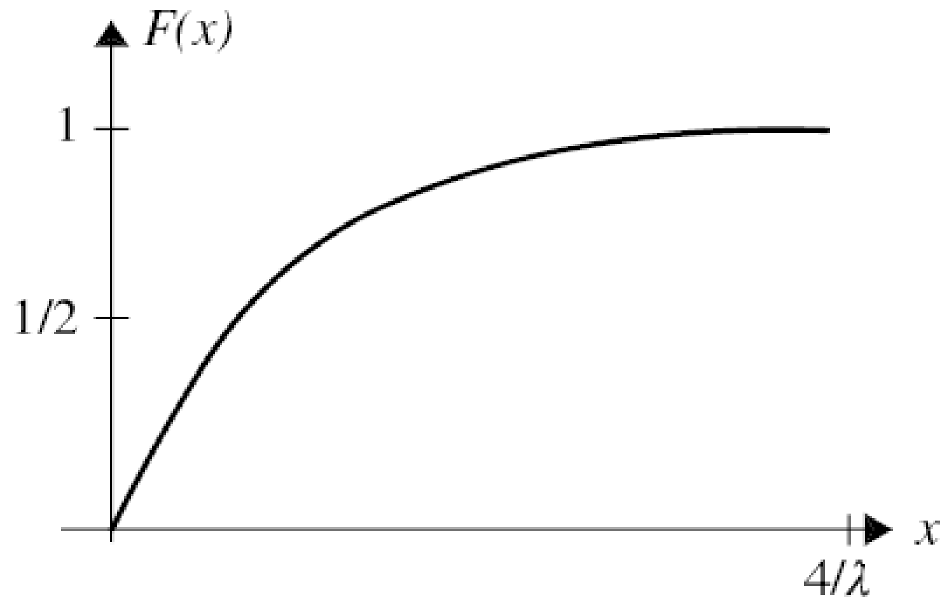


Figure 7.11 Exponential distribution function.

- An important feature of exponential distribution is its *memoryless property.*
- A nonnegative random variable *X* is called **memoryless** if, for all *s*, *t* ≥ 0,
- is the only continuous distribution which possesses a memoryless property.

Example 7.12

- The lifetime of a TV tube (in years) is an exponential random variable with mean 10.
- If Jim bought his TV set 10 years ago, what is the probability that its tube will last another 10 years?

1. Let X be the lifetime of the tube. Since X is an exponential random variable, there is no deterioration with age of the tube. Hence,

$$P(X > 20 \mid X > 10) = \qquad \approx 0.37$$

Relationship between Exponential and Geometric

- Sometimes exponential is considered to be the continuous analog of
- Exponential is the only memoryless continuous distribution, and is the only memoryless discrete distribution.

7.4 Gamma distributions

- Let $\{N(t): t \geq 0\}$ be a Poisson process, X_1 be the time of the first event, and for $n \geq 2$, let X_n be the time between the $(n-1)$st and nth events.
- $\{X_1, X_2, X_3, \ldots\}$ is a sequence of identically distributed exponential random variables with mean , where λ is the rate of $\{N(t): t \geq 0\}$.
- For this Poisson process let X be the time of the nth event. Then X is said to have a **distribution** with parameters (n, λ).

- is the time we will wait for the first event to occur, and is the time we will wait for the *n*th event to occur.
- Clearly, a gamma distribution with parameters $(1, \lambda)$ is identical with an distribution with parameter λ.

Distribution function

- Let X be a gamma random variable with parameters (n, λ).
- To find f, the density function of X, note that $\{X \leq t\}$ occurs if the time of the nth event is in $[0, t]$, that is, if the number of events occurring in $[0, t]$ is at least n.
- Hence F, the distribution function of X, is given by

$$F(t) = P(X \leq t) =$$

Density function

- The density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} \dfrac{(\lambda x)^{n-1}}{(n-1)!} & \text{if } x \geq 0 \\ 0 & elsewhere \end{cases}$$

is called the
density with parameters *(n, λ)*.

- Now we extend the definition of the gamma density from parameters *(n, λ)* to *(r, λ)*, where $r > 0$ is not necessarily a positive integer.
- In the formula of the gamma density function, the term is defined only for positive integers.
- So the only obstacle in such an extension is to find a function of *r* that has the basic property of the factorial function, namely,
  - $n! =$, and
  - coincides with $(n - 1)!$ when *n* is a positive integer.

- The function with these properties is : $(0,\infty) \rightarrow \mathbf{R}$ defined by

  $\Gamma(r) =$

- Γ(r + 1) is the natural generalization of *n*! for a noninteger $r > 0$.

Definition

A random variable X with probability density function

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{\Gamma(r)} & if\ x \geq 0 \\ 0 & elsewhere \end{cases}$$

is said to have a distribution with parameters (r, λ), λ > 0, r > 0.

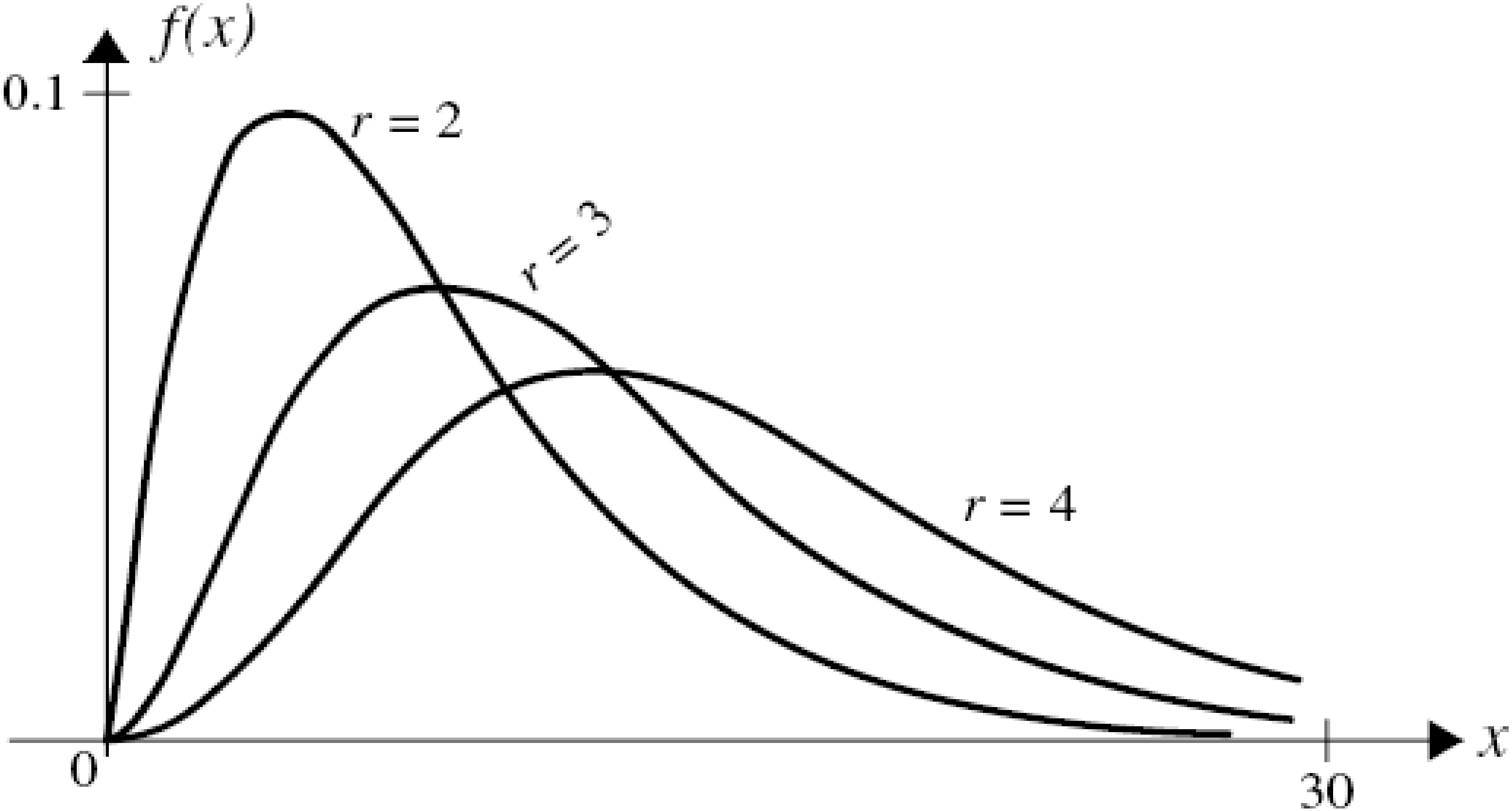


Figure 7.12 Gamma densities for $\lambda = 1/4$.

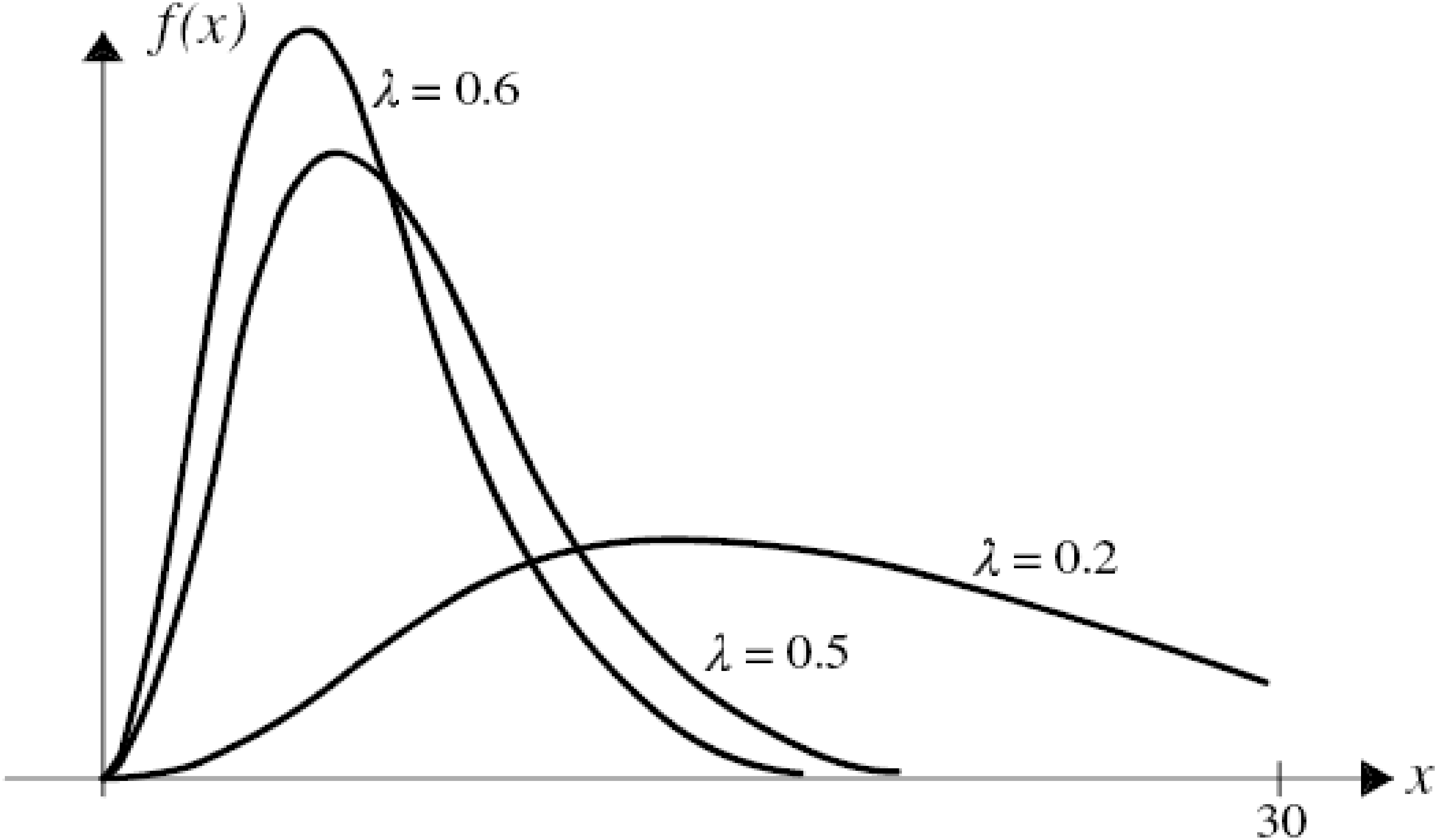


Figure 7.13 Gamma densities for $r = 4$.

Example 7.14

- Suppose that, on average, the number of β-particles emitted from a radioactive substance is four every second.
- What is the probability that it takes at least 2 seconds before the next two β-particles are emitted?

1. Let *N(t)* denote the number of β-particles emitted from a radioactive substance in [0, *t*].
2. It is reasonable to assume that $\{N(t): t \geq 0\}$ is a Poisson process.

   $\lambda =$

3.

$$P(X \geq 2) = \qquad\qquad = \int_2^{\infty} 16xe^{-4x}\,dx$$

$$= [-4xe^{-4x}]_2^{\infty} - \int_2^{\infty} -4e^{-4x}\,dx = 8e^{-8} + e^{-8} \approx 0.003$$

- For a gamma random variable with parameters *r* and *λ*,

$$E(X) =$$

$$Var(X) =$$

$$\sigma_X = \frac{\sqrt{r}}{\lambda}$$

Example 7.15

- There are 100 questions in a test. Suppose that, for all $s > 0$ and $t > 0$, the event that it takes t minutes to answer one question is independent of the event that it takes s minutes to answer another one.
- If the time that it takes to answer a question is exponential with mean 1/2, find the distribution, the average time, and the standard deviation of the time it takes to do the entire test.

1. Let X be the time to answer a question and $N(t)$ the number of questions answered by time t .

   Then $\{N(t): t \geq 0\}$ is a Poisson process at the rate of $\lambda =$ per minute.

2. Therefore, the time that it takes to complete all the questions is gamma with parameters $(100, 2)$.

   The average time to finish the test is = 100/2 = 50 minutes with standard deviation

$$= \sqrt{100/4} = 5$$

Relationship between Gamma and Negative Binomial

- Sometimes gamma is viewed as the continuous analog of negative