Ch6: Continuous Random Variables

6.1 Probability density functions

- In the case of discrete random variables a very small change in t may cause relatively large changes in the values of F.
- In cases such as
  - *the lifetime of a random light bulb*,
  - *the arrival time of a train at a station*, and
  - *the weight of a random watermelon grown in a certain field*,

  where the set of possible values of X is uncountable, small changes in x produce correspondingly small changes in the distribution of X.
- In such cases we expect that F, the distribution function of X, will be a function.

Definition

- *Let X be a random variable. Suppose that there exists a nonnegative real-valued function f* : **R** $\rightarrow$ [0,∞) *such that for any subset of real numbers A that can be constructed from intervals by a countable number of set operations,*

$$P(X \in A) = \int_A .$$

- *Then X is called **absolutely continuous** or, in this book, for simplicity,* .
- *Therefore, whenever we say that X is continuous, we mean that it is absolutely continuous and hence satisfies* (6.2).
- *The function f is called the* , *or simply the **density function** of X.*

Probability density functions

Let be the density function of a random variable X with distribution function .

$(a) F(t) =$

$(b) \int_{-\infty}^{\infty} f(x)dx =$

(c) If f is continuous, then

Even if f is not continuous, still

for every x at which f is continuous.

(d) For real number $a \le b$, $P(a \le X \le b) =$

$(e) P(a < X < b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b) =$

- Property (d) states that the probability of X being between a and b is equal to the area under the graph of f from a to b. Letting $a = b$ in (d), we obtain

$$P(X = a) =$$

- This means that for any real number a, $P(X = a) = 0$. That is, the probability that a continuous random variable assumes a certain value is

- The area over an interval *I* under the graph of *f* represents the probability that the random variable *X* will belong to .
- The area under *f* to the left of a given point *t* is, the value of the distribution function of *X* at *t* .

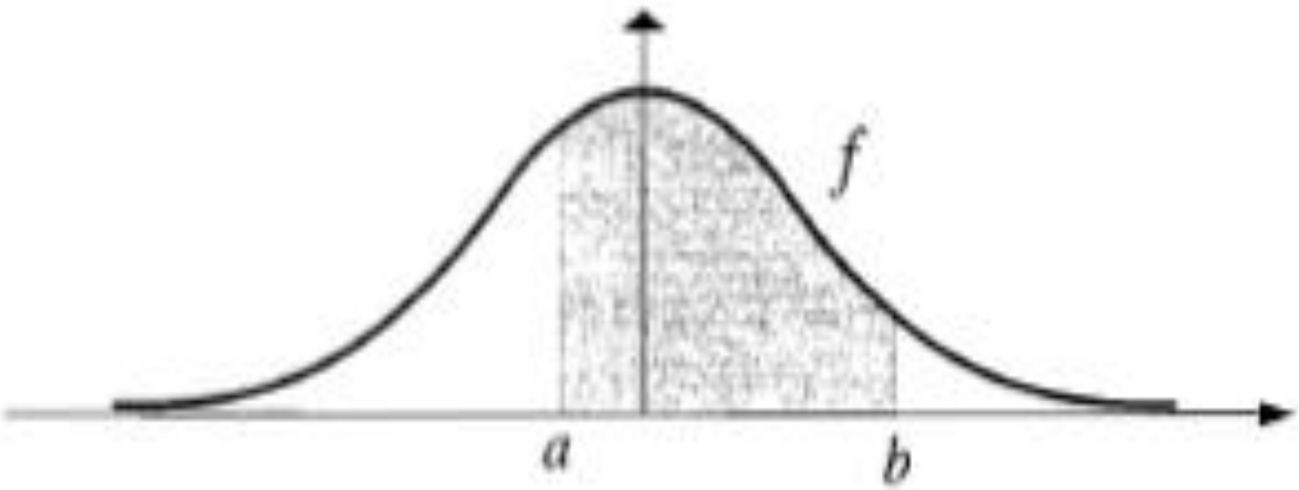

Figure 6.1 The shaded area under f is the probability that $X \in I = (a, b)$.

Example 6.1

- Experience has shown that while walking in a certain park, the time X, in minutes, between seeing two people smoking has a density function of the form

$$f(x) = \begin{cases} \lambda x e^{-x} & x > 0 \\ 0 & otherwise. \end{cases}$$

(a) Calculate the value of λ.

(b) Find the probability distribution function of X.

(c) What is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes? In at least 7 minutes?

Example 6.1

(a) We use the property $\int_{-\infty}^{\infty} f(x)dx =$:

$$\int_{-\infty}^{\infty} f(x)dx =$$

By integration by parts,

$$\int xe^{-x}dx = -(x+1)e^{-x}, \lambda\left[-(1+x)e^{-x}\right]_0^{\infty} = 1$$

But as $x -> \infty$,using l'Hoptial's rule, we get

$$\lim_{x\to\infty}(x+1)e^{-x} = \lim_{x\to\infty}\frac{(x+1)}{e^x} =$$

Therefore, it implies that

Example 6.1

(b) The distribution function of X, note F(t)=0 if t < 0.

For t >= 0,

$$F(t) = \qquad = \left[-(x+1)e^{-x}\right]_0^t = -(t+1)e^{-t} + 1$$

Thus, $F(t) = \begin{cases} 0 & t < 0 \\ & t \geq 0 \end{cases}$

(c) $P(2 < X < 5) = P(2 < X \leq 5) =$

$$= \qquad = (1-6e^{-5}) - (1-3e^{-2}) = 3e^{-2} - 6e^{-5} \approx 0.37$$

$$P(X \geq 7) = 1 - P(X < 7) = 1 - P(X \leq 7) =$$

Example 6.2

(a) Sketch the graph of the function

$$f(x)=\begin{cases}\dfrac{1}{2}-\dfrac{1}{4}|x-3| & 1\le x\le 5\\ 0 & otherwise,\end{cases}$$

and show that it is the probability density function of a random variable *X*.

(b) Find *F*, the distribution function of *X*, and show that it is continuous.

(c) Sketch the graph of *F*.

Example 6.2

(a) Note that

$$f(x) = \begin{cases} 0, & x < 1 \\ \\ 0, & x \geq 5 \end{cases}$$

The graph of f is as shown in next slide.

Now since f(x) >= 0 and the area under f from 1 to 5, being the area of the triangle ABC is (1/2)(4x1/2)=1, f is a density function of X.

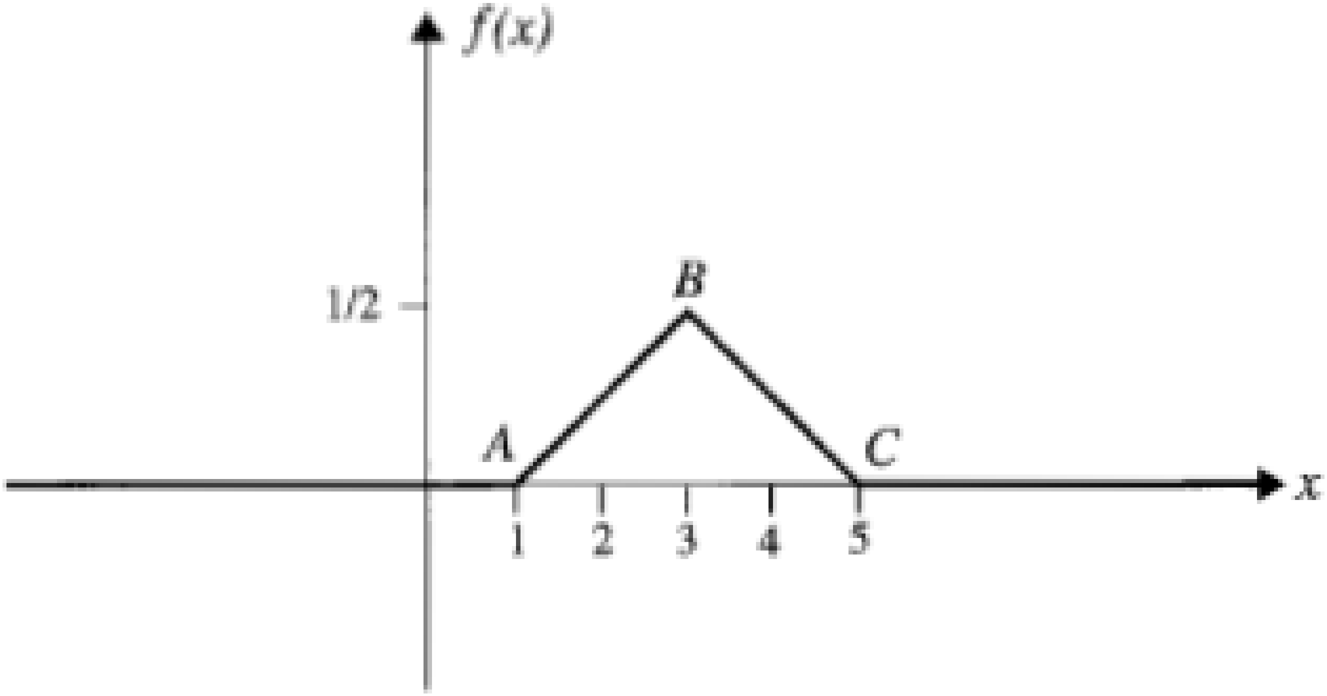

Figure 6.2 Density function of Example 6.2.

Example 6.2

(b) Using the formula $F(t)=\int_{-\infty}^{t} f(x)dx$

$$for \quad t<1, F(t)=$$

$$for \;\; 1\le t<3, F(t)=\int_{-\infty}^{t} f(x)dx= \qquad =\frac{1}{8}t^2-\frac{1}{4}t+\frac{1}{8}$$

$$for \;\; 3\le t<5, F(t)=$$

$$=-\frac{1}{8}t^2+\frac{5}{4}t-\frac{17}{8}$$

$$for \;\; t\ge 5, F(t)= \qquad =\frac{1}{2}+\frac{1}{2}=1$$

Example 6.2

F is continuous because

$$\lim_{t \to 1-} F(t) = \qquad ,$$

$$\lim_{t \to 3-} F(t) = \qquad = \frac{1}{2} = -\frac{1}{8}(3)^2 + \frac{5}{4}(3) - \frac{17}{8} = F(3),$$

$$\lim_{t \to 5-} F(t) = \qquad = F(5)$$

(c) See next slide.

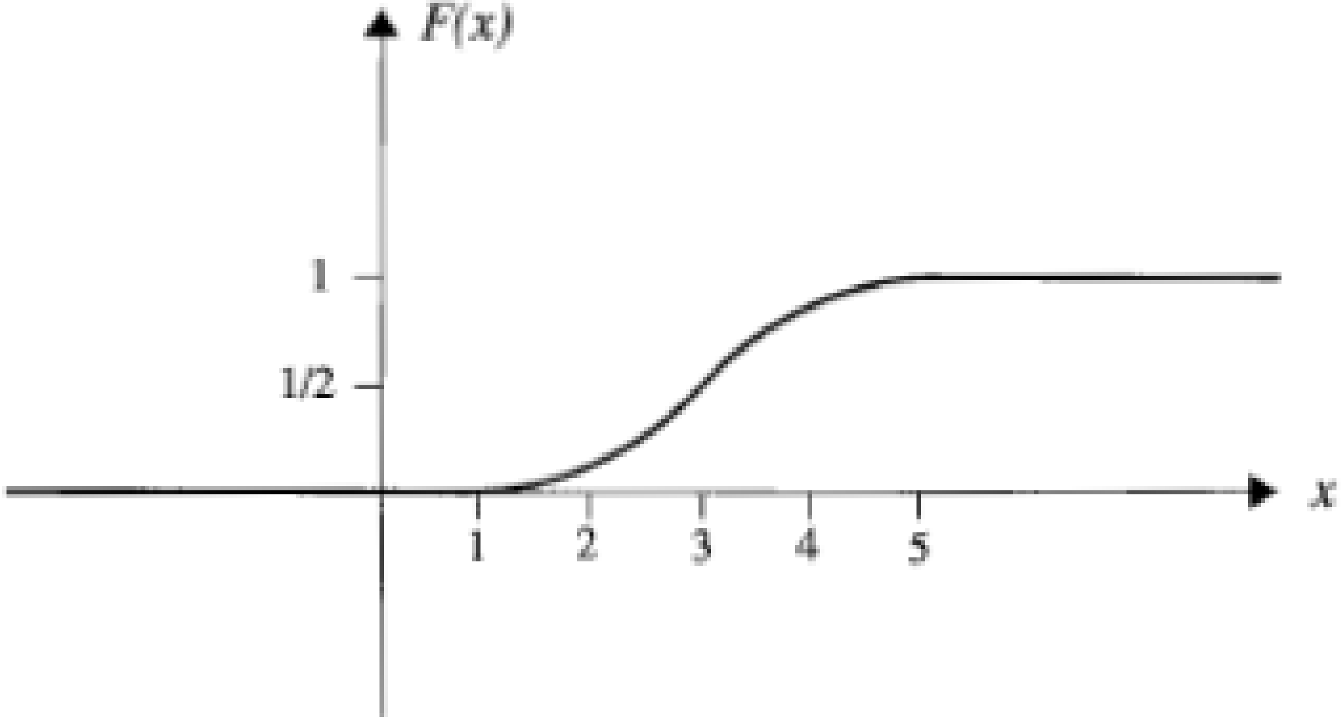

Figure 6.3 Distribution function of Example 6.2.

6.2 Density function of a function of a random variable

- f is the density function of a random variable X, then what is the density function of h(X)?

- Two methods:
  - **the method of** : to find the density of h(X) by calculating its distribution function;
  - **the method of** : to find the density of h(X) directly from the density function of X.

Example 6.3

- Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 2/x^2 & \text{if } 1 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the distribution and the density functions of $Y = X^2$.

Example 6.3

- 1. Let G and g be the distribution function and the density function of Y, respectively. By definition,

$$G(t) = \qquad = P(-\sqrt{t} \le X \le \sqrt{t})$$

$$= \begin{cases} 0 & t < 1 \\ & \qquad 1 \le t < 4 \\ 1 & t \ge 4 \end{cases}$$

$$P(1 \le X \le \sqrt{t}) = \qquad = 2 - \frac{2}{\sqrt{t}}$$

$$G(t) = \begin{cases} 0 & t < 1 \\ & \qquad 1 \le t < 4 \\ 1 & t \ge 4 \end{cases}$$

$The\ density\ function\ of\ Y, g, \ is:$

$$g(t) = \qquad = \begin{cases} & if\ 1 \le t \le 4 \\ 0 & elsewhere \end{cases}$$

Example 6.4

- Let X be a continuous random variable with distribution function F and probability density function f.
- In terms of f, find the distribution and the density functions of $Y = X^3$.

1. Let G and g be the distribution function and the density function of Y. Then

$$G(t) = P(Y \le t) = \qquad = P(X \le \sqrt[3]{t}) = F(\sqrt[3]{t})$$

$$Hence, \qquad g(t) = \qquad = \frac{1}{3\sqrt[3]{t^2}} f(\sqrt[3]{t})$$

Example 6.5

- The error of a measurement has the density function

$$f(x) = \begin{cases} 1/2 & if\ -1 < x < 1 \\ 0 & otherwise \end{cases}$$

Find the distribution and the density functions of the magnitude of the error.

Example 6.5

1. Let X be the error of the measurement. We want to find G, the distribution, and g, the density functions of |X|, the magnitude of the error. By definition,

$$G(t) = \qquad = \begin{cases} 0 & t < 0 \\ & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$

$$g(t) = \qquad = \begin{cases} & if\ 0 \le t \le 1 \\ 0 & elsewhere \end{cases}$$

6.3 Expectations and variances

- **Definition** *If X is a continuous random variable with probability density function f, the* ***expected value*** *of X is defined by*

$$E(X) =$$

Example 6.8

- In a group of adult males, the difference between the uric acid value and 6, the standard value, is a random variable *X* with the following probability density function:

$$f(x) = \begin{cases} \dfrac{27}{490}(3x^2 - 2x) & if \quad 2/3 < x < 3 \\ 0 & elsewhere. \end{cases}$$

Calculate the mean of these differences for the group.

Example 6.8

$$E(X) =$$

$$= \frac{27}{490}[\frac{3}{4}x^4 - \frac{2}{3}x^3]_{2/3}^{3} = \frac{283}{120} = 2.36$$

Remark 6.2

- If X is a continuous random variable with density function f, X is said to have a **finite expected value** if

that is, X has a finite expected value if the integral of $xf(x)$ converges absolutely. Otherwise, we say that the expected value of X is not finite.

Example 6.9

- A random variable *X* with density function

$$f(x) = \frac{c}{1+x^2}, \quad -\infty < x < \infty,$$

is called a **Cauchy random variable**.

(a) Find *c*.

(b) Show that *E(X)* does not exist.

Example 6.9

(a)

$$\Rightarrow \int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = c\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$\because \int \frac{dx}{1+x^2} = \arctan x$$

$$\therefore 1 = c\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \qquad = c\pi$$

$$c = {}^{1}\!/_{\pi}$$

(b)

$$\because \qquad = \int_{-\infty}^{\infty} \frac{|x|\,dx}{\pi(1+x^2)} = 2\int_{0}^{\infty} \frac{x\,dx}{\pi(1+x^2)}$$

$$=$$

$\therefore$ E(X) does not exist

Theorem 6.2

- *For any continuous random variable X with probability distribution function F and density function f,*

$$E(X) =$$

Theorem 6.2

- *Proof:*

  Note that

$$E(X)=\int_{-\infty}^{\infty} xf(x)dx=\int_{-\infty}^{0} xf(x)dx+\int_{0}^{\infty} xf(x)dx$$

$$=-\int_{-\infty}^{0}\left(\int_{0}^{-x} dt\right)f(x)dx+\int_{0}^{\infty}\left(\int_{0}^{x} dt\right)f(x)dx$$

$$=-\int_{0}^{\infty}\left(\int_{-\infty}^{-t} f(x)dx\right)dt+\int_{0}^{\infty}\left(\int_{t}^{\infty} f(x)dx\right)dt$$

Theorem 6.2

where the last equality is obtained by changing the order of integration. The theorem follows

since $\int_{-\infty}^{-t} f(x)dx = F(-t)$

and $\int_{t}^{\infty} f(x)dx = P(X > t) = 1 - F(t)$

Remark 6.4

- For any random variable *X*,

$$E(X) = \int_0^{\infty} P(X > t)dt - \int_0^{\infty} P(X \le -t)dt$$

- In particular, if *X* is nonnegative, that is, $P(X < 0) = 0$, this theorem states that

$$E(X) =$$

Theorem 6.3

- *Let X be a continuous random variable with probability density function f (x); then for any function h*: $\mathbf{R} \to \mathbf{R}$,

$$E[h(X)] =$$

(Theorem 4.2)

Corollary

- Let X be a continuous random variable with probability density function f (x). Let h_1, h_2, . . . , h_n be real-valued functions, and α_1, α_2, . . . , α_n be real numbers. Then $E[\alpha_1 h_1(X) + \alpha_2 h_2(X) + \cdot\cdot\cdot + \alpha_n h_n(X)] =$

- This corollary implies that if *α* and *β* are constants, then $E(\alpha X + \beta) =$

Example 6.10

- A point X is selected from the interval (0, π/4) randomly. Calculate E(cos 2X) and E($\cos^2$ X).

1. Calculate the distribution function of *X:*

$$F(t) = P(X \le t) = \begin{cases} 0 & t < 0 \\ & 0 \le t < \dfrac{\pi}{4} \\ 1 & t \ge \dfrac{\pi}{4} \end{cases}$$

Example 6.10

2. Thus f, the probability density function of X, is

$$f(t) = \begin{cases} \\ 0 & \text{otherwise} \end{cases}$$

3.

$$E(\cos 2X) = \qquad\qquad = [\frac{2}{\pi}\sin 2x]_0^{\pi/4} = \frac{2}{\pi}$$

$$\because \cos^2 X = (1 + \cos 2X)/2$$

$$\therefore E(\cos^2 X) = \qquad\qquad = \frac{1}{2} + \frac{1}{\pi}$$

Variances of Continuous Random Variables

- **Definition** *If X is a continuous random variable with E(X) = μ, then* Var*(X) and* σ_X*, called the* ***variance*** *and* ***standard deviation*** *of X, respectively, are defined by*

$$Var(X) =$$

$$\sigma_X =$$

- Therefore, if *f* is the density function of *X*, then by Theorem 6.3,

$$Var(X) =$$

- Var(X) =
- Var(aX + b) =

σ_{aX+b} =

a and *b* being constants.

Example 6.11

- The time elapsed, in minutes, between the placement of an order of pizza and its delivery is random with the density function

$$f(x) = \begin{cases} 1/15 & if \;\; 25 < x < 40 \\ 0 & otherwise. \end{cases}$$

(a) Determine the mean and standard deviation of the time it takes for the pizza shop to deliver pizza.

(b) Suppose that it takes 12 minutes for the pizza shop to bake pizza. Determine the mean and standard deviation of the time it takes for the delivery person to deliver pizza.

Example 6.11

(a) Let the time elapsed between the placement of an order and its delivery be X minutes.

$$E(X) = \qquad = 32.5, \qquad E(X^2) = \int_{25}^{40} x^2 \frac{1}{15} dx = 1075$$

$$\therefore Var(X) = \qquad = 18.75$$

$$\sigma_X = \qquad = 4.33$$

(b) The time it takes for the delivery person to deliver pizza is $X - 12$.

$$E(X - 12) = \qquad = 32.5 - 12 = 20.5$$

$$\sigma_{X-12} = \qquad = \sigma_X = 4.33$$

Remark 6.5

- For a continuous random variable X, the moments, absolute moments, moments about a constant c, and central moments are all defined in a manner similar to those of Section 4.5 (the discrete case).