Ch5: Special Discrete Distributions

5.1 Bernoulli and binomial random variables

- The sample space of a Bernoulli trial contains two points, *s* and *f* .
- The random variable defined by X(s) = 1 and X(f) = 0 is called a **Bernoulli random variable.**
- If p is the probability of a success, then $1 - p$ (sometimes denoted q) is the probability of a failure.
- Hence the probability mass function of X is

$$p(x) =$$

Bernoulli random variable

- $E(X) =$
- $E(X^2) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = p,$
- $Var(X) =$

$$\sigma_X = \sqrt{p(1-p)}$$

Example 5.1

- If in a throw of a fair die the event of obtaining 4 or 6 is called a success, and the event of obtaining 1, 2, 3, or 5 is called a failure, then it is a Bernoulli trial.
- Get the probability mass function, expected value, and variance.

1. p = 1/3
2. $p(x) =$
3. E(X) = p = 1/3, Var(X)=p(1-p)=1/3(1-1/3)=2/9

- Let X_1, X_2, X_3, … be a sequence of Bernoulli random variables.
- If, for all j_i=0 or 1, the sequence of events $\{X_1=j_1\}$, $\{X_2=j_2\}$, $\{X_3=j_3\}$, … are independent, we say that $\{X_1, X_2, X_3, \ldots\}$ and the corresponding Bernoulli trials are

Binomial random variable

- If n Bernoulli trials all with probability of success p are performed independently, then X, the number of successes, is called a binomial with parameters n and p. The set of possible values of X is {0, 1, 2, . . . , *n*}.
- We write as in short.

Binomial random variable

- **Theorem 5.1** *Let X be a binomial random variable with parameters n and p. Then p(x), the probability mass function of X, is*

$$p(x) = P(X = x) = \begin{cases} & \text{if } x = 0, 1, 2, \ldots, n \\ 0 & \text{elsewhere.} \end{cases}$$

- **Definition** *The function p(x) given above is called the **binomial** probability mass function with parameters (n, p).*

Example 5.3

- In a county hospital 10 babies, of whom six were boys, were born last Thursday.
- What is the probability that the first six births were all boys? Assume that the events that a child born is a girl or is a boy are equiprobable.

1. Let A be the event that the first six births were all boys and the last four all girls. Let X be the number of boys; then X is binomial with parameters 10 and 1/2.
2. $P(A \mid X = 6) =$

$$= \frac{(\frac{1}{2})^{10}}{\binom{10}{6}(\frac{1}{2})^{6}(\frac{1}{2})^{4}} = \frac{1}{\binom{10}{6}} \approx 0.0048$$

Example 5.4

- In a small town, out of 12 accidents that occurred in June 1986, four happened on Friday the 13th.
- Is this a good reason for a superstitious person to argue that Friday the 13th is inauspicious?

1. Suppose the probability that each accident occurs on Friday the 13th is 1/30.
2. the probability of at least four accidents on Friday the 13th is
3. Since the probability of four or more of these accidents occurring on Friday the 13th is very small, this is a good reason for superstitious persons to argue that Friday the 13th is inauspicious.

- Let X be a binomial random variable with parameters (n, p), $0 < p < 1$, and probability mass function $p(x)$.
- We will now find the value of X at which $p(x)$ is maximum.
- For any real number t, let $[t]$ denote the largest integer less than or equal to t. We will prove that $p(x)$ is maximum at $x =$
To do so, we note that

$$\frac{p(x)}{p(x-1)} =$$

$$= \frac{(n-x+1)p}{x(1-p)}$$

$$= \frac{(n+1)p - xp + x - x}{x(1-p)} = \frac{[(n+1)p - x] + x(1-p)}{x(1-p)}$$

$$=$$

- This equality shows that $p(x) > p(x-1)$ if and only if , or, equivalently, if and only if
- Hence as x changes from 0 to $[(n + 1)p]$, $p(x)$.
- As x changes from $[(n + 1)p]$ to n, $p(x)$.
- The maximum value of $p(x)$ [the peak of the graphical representation of $p(x)$] occurs at .
- Figure 5.1 shows binomial random variables with parameters (5, 1/2), (10, 1/2), and (20, 1/2).

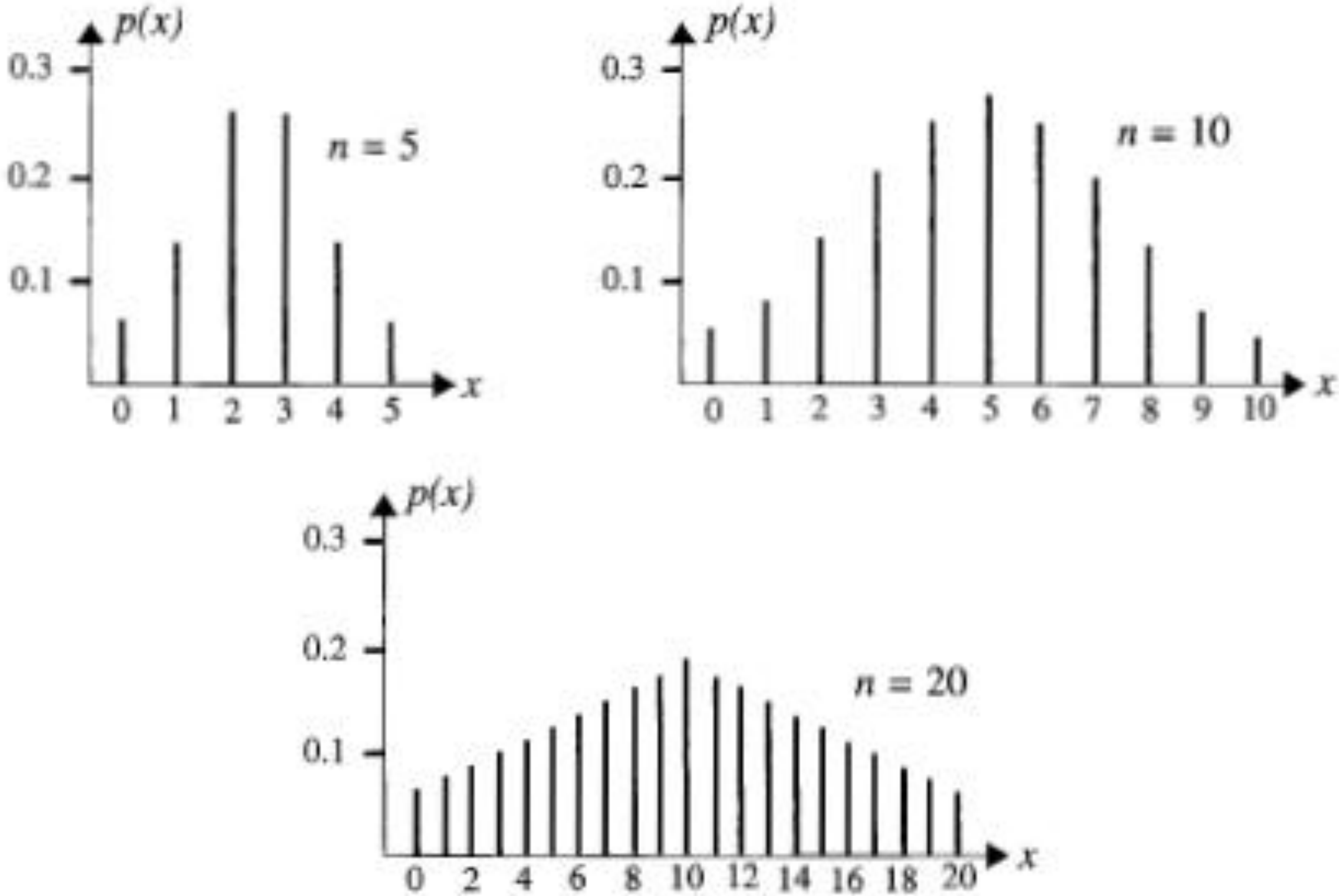

Figure 5.1 Examples of binomial probability mass functions.

Expectations and Variances of Binomial Random Variables

$$E(X) =$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} = np \sum_{i=0}^{n-1} \binom{n-1}{i} p^{i} (1-p)^{(n-1)-i}$$

$$=$$

- E(X) = np, Var(X) = , $\sigma_X = \sqrt{np(1-p)}$

5.2 Poisson random variable

- In many cases, direct calculation of *p(x)*

$$p(x) =$$

for binomial random variable is not possible, because even for moderate values of n, $n!$ exceeds the largest integer that a computer can store.

Poisson random variable

In 1837 French mathematician Simeon-Denis Poisson introduced the following procedure to obtain the formula that approximates p(x) when $n \to \infty,\ p \to 0,\ np = \lambda$ remains a fixed quantity of moderate value.

$$P(X = i) =$$

$$= \underbrace{\frac{n(n-1)(n-2)\cdots(n-i+1)}{n^i}}_{\to 1} \frac{\lambda^i}{i!} \frac{\overbrace{(1-\frac{\lambda}{n})^n}^{\to e^{-\lambda}}}{\underbrace{(1-\frac{\lambda}{n})^i}_{\to 1}}$$

for large n and appreciable λ

$$P(X = i) \to$$

Poisson random variable

- German-Russian mathematician L. V. Bortkiewicz demonstrated its significance for both in probability theory and its applications.
- Among other things, Bortkiewicz argued that since

$$\sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^{i}}{i!} =$$

Poisson's approximation by itself is a probability mass function.

Definition

- *A discrete random variable X with possible values* 0, 1, 2, 3, . . . *is called* ***Poisson*** *with parameter* λ, $\lambda > 0$, *if*

$$P(X = i) = \qquad\qquad , \quad i = 0, 1, 2, 3, \cdots$$

$$E(X) =$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E(X^2) =$$

$$Var(X) =$$

$$\sigma_X = \sqrt{\lambda}$$

- Under the conditions specified in our discussion, probabilities can be approximated by probabilities.
- Such approximations are generally good if $p < 0.1$ and $np \leq 10$.
- If $np > 10$, it would be more appropriate to use normal approximation, discussed in Section 7.2.

Poisson Example

- Let X be the number of misprints on a document page typed by a secretary.
- Then X is a random variable if a word is called a success, provided that it is misprinted!
- Since misprints are rare events, the number of words is large, and , the average number of misprints, is of moderate value, X is approximately a Poisson random variable.

Example 5.11

- Suppose that, on average, in every three pages of a book there is one typographical error. If the number of typographical errors on a single page of the book is a Poisson random variable, what is the probability of at least one error on a specific page of the book?

1. Let X be the number of errors on the page we are interested in. Then X is a Poisson random variable with E(X) = 1/3.
2. $$P(X = n) =$$
3. $$P(X \geq 1) =$$

5.3 Other discrete random variables

- **Geometric Random Variables**:
  - suppose that a sequence of independent Bernoulli trials, each with probability of success p, $0 < p < 1$, are performed.
  - Let X be the number of experiments until the first success occurs. Then X is a discrete random variable called .
  - It is defined on S, its set of possible values is $\{1, 2, \ldots\}$, and

$$P(X = n) = \qquad\qquad , n = 1, 2, 3, \ldots .$$

- **Definition** *The probability mass function*

$$p(x) =$$

*is called **geometric**.*

- $$E(X) = \frac{1}{p}, Var(X) = \frac{1-p}{p^2}, \sigma_X = \frac{\sqrt{1-p}}{p}$$

Expected Value of Geometric

- Find the expected value of a geometric random variable.
- **Solution** With q = 1 - p we have that

$$E[X] = \sum_{n=1}^{\infty} nq^{n-1}p = \mathrm{p}\sum_{\mathrm{n}=0}^{\infty} \frac{d}{dq}(q^{n})$$

$$= \mathrm{p}\frac{\mathrm{d}}{\mathrm{dq}}\left(\sum_{n=0}^{\infty} q^{n}\right) = \mathrm{p}\frac{\mathrm{d}}{\mathrm{dq}}\left(\frac{1}{1-q}\right)$$

$$= \frac{\mathrm{p}}{(1\text{-}\mathrm{q})^{2}} = \frac{1}{\mathrm{p}}$$

Variance of Geometric

- Find the variance value of a geometric random variable.
- **Solution** To determine Var(X) let us first compute $E[X^2]$. With q = 1 – p ,

$$E[X^2] = \sum_{n=1}^{\infty} n^2 q^{n-1} p = \mathrm{p}\sum_{n=1}^{\infty} \frac{d}{dq}(nq^n) = \mathrm{p}\frac{\mathrm{d}}{\mathrm{dq}}\left(\sum_{n=1}^{\infty} nq^n\right) = \mathrm{p}\frac{\mathrm{d}}{\mathrm{dq}}\left(\frac{q}{1-q}E[X]\right)$$

$$= \mathrm{p}\frac{\mathrm{d}}{\mathrm{dq}}[q(1-q)^{-2}] = p\left[\frac{1}{p^2} + \frac{2(1-p)}{p^3}\right] = \frac{2}{p^2} - \frac{1}{p}$$

- Hence, since E[X] = 1/p ,

$$Var(X) = \frac{1-p}{p^2}$$

Example 5.18

- From an ordinary deck of 52 cards we draw cards at random, *with replacement*, and successively until an ace is drawn. What is the probability that at least 10 draws are needed?

1. Let X be the number of draws until the first ace.
2. X is geometric with parameter $p = 1/13$.
3. $$P(X \geq 10) =$$
   $$= \frac{1}{13} \cdot \frac{(12/13)^9}{1 - 12/13} = \left(\frac{12}{13}\right)^9 \approx 0.49$$

Remark

- There is a shortcut to the solution of this problem:
  - The probability that at least 10 draws are needed to get an ace is the same as the probability that in the first nine draws there are no aces.
  - This is equal to

Memoryless property

- Let X be a geometric random variable with parameter p, $0 < p < 1$. Then, for all positive integers n and m,

$$P(X > n+m \mid X > m) = \frac{P(X > n+m)}{P(X > m)} =$$

- In successive independent Bernoulli trials, the probability that the next n outcomes are all failures does not change if we are given that the previous m successive outcomes were all failures.

- Geometric random variable is the only memoryless discrete random variable in the following sense.
- Let X be a discrete random variable with the set of possible values $\{1, 2, 3 \ldots\}$. If for all positive integers n and m,

  $$P(X > n + m \mid X > m) =$$

  then X is a geometric random variable. That is, there exists a number p, $0 < p < 1$, such that

  $$P(X = n) = \qquad\qquad , n \geq 1.$$

Example 5.19

- A father asks his sons to cut their backyard lawn. Since he does not specify which of the three sons is to do the job, each boy tosses a coin to determine the odd person, who must then cut the lawn.
- In the case that all three get heads or tails, they continue tossing until they reach a decision. Let p be the probability of heads and $q = 1 - p$, the probability of tails.

  (a) Find the probability that they reach a decision in less than n tosses.

  (b) If $p = 1/2$, what is the minimum number of tosses required to reach a decision with probability 0.95?

Example 5.19

(a)

1. The probability that they reach a decision on a certain round of coin tossing is

2. Let X be the number of tosses until the reach a decision; X is a geometric random variable with parameter *3pq*.

$$P(X < n) =$$

(b)

1. Find the minimum n so that $P(X \le n) \ge 0.95$. This gives $P(X > n) \le 0.05$. But $P(X > n) = (1-3pq)^n = (1-3/4)^n = (1/4)^n$
2. Therefore, This gives n >= 2.16; hence the smallest n is 3.

Negative Binomial Random Variables

- Suppose that a sequence of independent Bernoulli trials, each with probability of success p, $0 < p < 1$, is performed.
- Let X be the number of experiments until the r-th success occurs.
- Then X is a discrete random variable called a **negative binomial.** Its set of possible values is $\{r, r+1, r+2, r+3, \ldots\}$ and

$$P(X = n) =$$

- **Definition** *The probability mass function*

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, 0 < p < 1, x = r, r+1, r+2, \ldots$$

is called ***negative binomial*** *with parameters (r, p).*

- $E(X) =$ $\quad, Var(X) =$ $\quad, \sigma_X = \dfrac{\sqrt{r(1-p)}}{p}$

Expected Value of Negative Binomial

- Compute the expected value of a negative binomial random variable with parameters r and p.
- **Solution**

$$E[X^k] = \sum_{n=r}^{\infty} n^k \binom{n-1}{r-1} p^r (1-p)^{n-r} = \frac{\mathrm{r}}{\mathrm{p}} \sum_{n=r}^{\infty} n^{k-1} \binom{n}{r} p^{r+1} (1-p)^{n-r} \quad \text{since} \quad \mathrm{n}\binom{\mathrm{n-1}}{\mathrm{r-1}} = r\binom{n}{r}$$

$$= \frac{\mathrm{r}}{\mathrm{p}} \sum_{m=r+1}^{\infty} (m-1)^{k-1} \binom{m-1}{r} p^{r+1} (1-p)^{m-(r+1)} \quad \text{by setting } \mathrm{m = n+1}$$

$$= \frac{\mathrm{r}}{\mathrm{p}} E[(Y-1)^{k-1}]$$

- **where Y is a negative binomial random variable with parameters r+1, p. Setting k=1 in the preceding equation yields**

$$E[X] = \frac{r}{p}$$

Variance of Negative Binomial

- Compute the variance of a negative binomial random variable with parameters r and p.
- **Solution** Setting k = 2 in the preceding equation, and using the formula for the expected value of a negative binomial random variable, gives that

$$E[X^2] = \frac{r}{p} E[Y-1] = \frac{\mathrm{r}}{\mathrm{p}} \left(\frac{r+1}{p} - 1 \right)$$

- Hence, since E[X] = 1/p ,

$$Var(X) = \frac{r}{p} \left(\frac{r+1}{p} - 1 \right) - \left(\frac{r}{p} \right)^2 = \frac{\mathrm{r}(1-\mathrm{p})}{p^2}$$

Example 5.20

- Sharon and Ann play a series of backgammon games until one of them wins five games. Suppose that the games are independent and the probability that Sharon wins a game is 0.58.

  (a) Find the probability that the series ends in seven games.

  (b) If the series ends in seven games, what is the probability that Sharon wins?

Example 5.20

(a) Let X be the number of games until Sharon wins five games. Let Y be the number of games until Ann wins five games. X and Y are negative binomial with parameters (5, 0.58) and (5, 0.42). The probability that the series ends in seven games is

$$P(X = 7) + P(Y = 7) =$$

(b) Let A be the event that Sharon wins and B be the event that the series ends in seven games. Then the desired probability is

$$P(A \mid B) = \frac{P(AB)}{P(\mathrm{B})} =$$

Example 5.21 (Attrition Ruin Problem)

- Two gamblers play a game in which in each play gambler A beats B with probability p, $0 < p < 1$, and loses to B with probability $q = 1 - p$.
- Suppose that each play results in a forfeiture of $1 for the loser and in no change for the winner.
- If player A initially has a dollars and player B has b dollars, what is the probability that B will be ruined?

1. Let E_i be the event that, in the first b+i plays, B loses b times. Let A^* be the event that A wins.

$$P(A^*) =$$

2. If every time that A wins is called a success, E_i is the event that the *b* th success occurs on the *(b + i)* th play.

$$P(E_i) =$$

3. Therefore,

$$P(A^*) =$$

Hypergeometric Random Variables

- Suppose that, from a box containing D defective and $N - D$ nondefective items, n are drawn at random and *without replacement.*
- Furthermore, suppose that the number of items drawn does not exceed the number of defective or the number of nondefective items. That is, suppose that $n \leq \min(D, N-D)$.

Hypergeometric Random Variables (Cont.)

- Let *X* be the number of defective items drawn. Then *X* is a discrete random variable with the set of possible values {0, 1, . . . *n*}, and a probability mass function

$$p(x) = P(X = x) = \qquad\qquad , \quad \mathrm{x} = 0, 1, 2, \ldots, \mathrm{n}$$

- **Definition** *Let N, D, and n be positive integers with n ≤* min*(D,N − D). Then*

$$p(x) = P(X = x) = \begin{cases} \dfrac{\dbinom{D}{x}\dbinom{N-D}{n-x}}{\dbinom{N}{n}} & if\ \ x \in \{0,1,2,\ldots,n\} \\ 0 & elsewhere \end{cases}$$

*is said to be a **hypergeometric** probability mass function*

$$E(X) = \qquad\qquad , \quad Var(X) =$$

Expected Value of Hypergeometric

- Determine the expected value of X, a hypergeometric random variable with parameters n, N, D.
- **Solution**

$$E[X^k] = \sum_{i=0}^{n} i^k p\{X = i\}$$

$$= \sum_{i=1}^{n} i^k \binom{D}{i}\binom{N-D}{n-i} \Big/ \binom{N}{n}$$

- **Using the identities**

$$\mathrm{i}\binom{\mathrm{D}}{\mathrm{i}} = \mathrm{D}\binom{\mathrm{D-1}}{\mathrm{i-1}} \quad \text{and} \quad \mathrm{n}\binom{\mathrm{N}}{\mathrm{n}} = \mathrm{N}\binom{\mathrm{N-1}}{\mathrm{n-1}}$$

Expected Value of Hypergeometric

- We obtain that

$$E[X^k] = \frac{nD}{N} \sum_{i=1}^{n} i^{k-1} \binom{D-1}{i-1} \binom{N-D}{n-i} \Big/ \binom{N-1}{n-1}$$

$$= \frac{nD}{N} \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{D-1}{j} \binom{N-D}{n-1-j} \Big/ \binom{N-1}{n-1}$$

$$= \frac{\mathrm{nD}}{\mathrm{N}} E[(Y+1)^{k-1}]$$

- Where Y is a hypergeometric random variable with parameters n-1, N-1, D-1. Hence, upon setting k=1 we see that

$$E[X] = \frac{nD}{N}$$

Variance of Hypergeometric

- Determine the variance of X, a hypergeometric random variable with parameters n, N, D.
- **Solution**

$$E[X^2] = \frac{nD}{N} E[Y+1]$$

$$= \frac{\mathrm{nD}}{\mathrm{N}} \left[\frac{(n-1)(D-1)}{N-1} + 1 \right]$$

- As E[X] = nD/N we can conclude that

$$Var(X) = \frac{nD}{N} \left[\frac{(n-1)(D-1)}{N-1} + 1 - \frac{nD}{N} \right]$$

Variance of Hypergeometric

- If we let p=D/N denote the fraction of items that are defective, then it follows from above equation, after a little algebra, that

$$Var(X) = \frac{N-n}{N-1} np(1-p)$$

Example 5.23

- In 500 independent calculations a scientist has made 25 errors. If a second scientist checks seven of these calculations randomly, what is the probability that he detects two errors? Assume that the second scientist will definitely find the error of a false calculation.

1. Let X be the number of errors found by the second scientist. Then X is hypergeometric with $N = 500$, $D = 25$, and $n = 7$.
2.

Example 5.24

- In a community of $a + b$ potential voters, a are for abortion and b $(b < a)$ are against it.
- Suppose that a vote is taken to determine the will of the majority with regard to legalizing abortion.
- If n $(n < b)$ random persons of these $a + b$ potential voters do not vote, what is the probability that those against abortion will win?

1. Let X be the number of those who do not vote and are for abortion. The persons against abortion will win if and only if

2.

$$P(X > \frac{a-b+n}{2}) =$$

Remark 5.2

- If the n items that are selected at random from the D defective and $N-D$ nondefective items are chosen *with replacement* rather than *without replacement*, then X, the number of defective items, is a binomial random variable with parameters n and D/N. Thus

$$P(X = x) =$$

Remark 5.2 (Cont.)

- Now, if *N* is very large, it is not that important whether a sample is taken with or without replacement.
- Therefore, for large *N*, the binomial probability mass function is an excellent approximation for the hypergeometric probability mass function, which can be stated mathematically as follows.

$$\lim_{\substack{N \to \infty \\ D \to \infty \\ D/N \to P}} \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} =$$