Ch 3 Conditional Prob. and Independence

便箋標題 2009/9/22

Def:

$$p(A|B) = \frac{p(AB)}{p(B)} \qquad p(B) > 0$$

Ex 3.2 From the set of all families with two children,

a family is selected at random and is found to have a girl.

What is the prob. that the other child of the family is a girl?

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

F, F
M, M
M, F
F, M

3.2 Law of multiplication

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$P(AB) = P(B)P(A|B)$$

$$P(AB) = P(BA) = P(A)P(B|A)$$

Ex 3.9

Suppose that five good fuses and two defective ones have been mixed up

To find the defective fuses, test one-by-one at random without replacement

prob. that we find both of the defective fuses in the first two tests?

$$P(D_1 D_2) = P(D_1)\, P(D_2 \mid D_1)$$

$$= \frac{2}{7} \times \frac{1}{6} = \frac{1}{21}$$

$$P(ABC) = P(A)\, P(B \mid A)\, P(C \mid AB)$$

Thm 3.2

$$p(A_1 A_2 \cdots A_n) = p(A_1)\,p(A_2 | A_1)\,p(A_3 | A_1 A_2) \cdots p(A_n | A_1 A_2 \cdots A_{n-1})$$

Law of multiplication

3.3 Law of total Probability

Thm 3.3 $$p(A) = p(A | B)\,p(B) + p(A | B^c)\,p(B^c)$$

$$p(A) = p(AB) + p(AB^c)$$

where

$$p(AB) = p(B)\,p(A|B)$$
$$p(AB^c) = p(B^c)\,p(A|B^c)$$

Ex 3.14 (Gambler's Ruin Problem)

Two gamblers play "heads or tails"

in which each time a fair coin lands heads up → A wins \$1 from B

lands tails up → B wins \$1 from A

Suppose player A with a dollars

B with b dollars initially

What is the prob. that

(a) A will be ruined (b) the game goes forever with nobody winning?

(a) Let E be the event that A will be ruined if he/she starts with i dollars, and let

$$P_i = p(E)$$

Define F : the event that A wins the first game

$$p(E) = \underbrace{p(E|F)}_{P_{i+1}} p(F) + \underbrace{p(E|F^c)}_{P_{i-1}} p(F^c)$$

$$p(E) = P_{i+1} \cdot \frac{1}{2} + P_{i-1} \cdot \frac{1}{2} \quad \therefore P_i = \frac{1}{2} P_{i+1} + \frac{1}{2} P_{i-1}$$

$P_0 = 1$, $P_{a+b} = 0$

$P_{i+1} - P_i = P_i - P_{i-1}$

Let $P_1 - P_0 = \alpha$

$P_{i+1} - P_i = P_i - P_{i-1} = \cdots P_1 - P_0 = \alpha$

$P_1 = P_0 + \alpha$

$P_2 = P_1 + \alpha = P_0 + 2\alpha$

$\therefore P_i = P_0 + i\alpha = 1 + i\alpha$

$P_{a+b} = 0$

$\therefore 0 = 1 + (a+b)\alpha \qquad \therefore \alpha = \frac{-1}{a+b}$

$\therefore p_i = 1 - \frac{i}{a+b} = \frac{a+b-i}{a+b}$

(b) q_i : B will be ruined if he or she starts with i dollars.

$q_i = \frac{a+b-i}{a+b}$

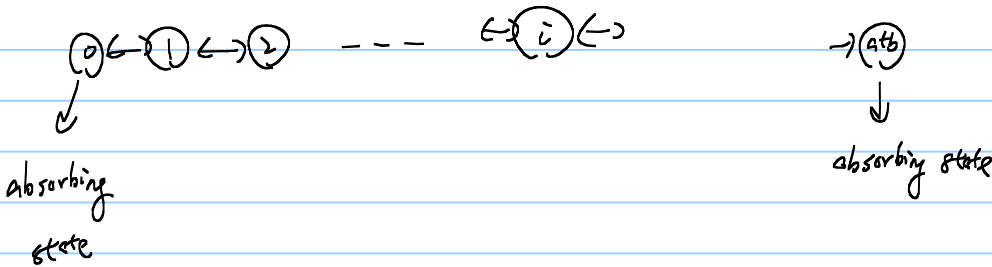
The prob. that the game goes forever with nobody winning is

$1 - q_b - p_a = 1 - \frac{a+b-b}{a+b} - \frac{a+b-a}{a+b} = \frac{a+b}{a+b} - \frac{a}{a+b} - \frac{b}{a+b}$

$$= \frac{0}{a+b} = 0$$

$\Rightarrow$ If this game is played successively,

eventually either A or B is ruined.



Law of Total prob.

$\{B_1, B_2, \cdots B_n\}$ be a set of nonempty subsets of the sample space S.

If $B_1, B_2, \cdots B_n$ are mutually exclusive and $\bigcup_{i=1}^{n} B_i = S$

$\{B_1, B_2, \cdots B_n\}$ is called a partition of S.

Thm 3.4 : let $\{B_1, B_2, \cdots B_\infty\}$ be a sequence of mutually exclusive events of S such that $\bigcup_{i=1}^{\infty} B_i = S$

$$P(A) = \sum_{i=1}^{\infty} P(A \mid B_i)\, P(B_i)$$

Ex 3-17

An urn contains 10 white and 12 red chips.

Two chips are drawn at random and, without looking at their colors, are discards, what is the prob. that a third chip drawn is red?

$$\frac{12}{22}$$

Ans: Let R_i be the event that the i-th chip drawn is red

W_i: " " " " is white.

$\{R_2W_1, W_2R_1, R_2R_1, W_2W_1\}$ is a partition of S.

$$p(R_3) = p(R_3 \mid R_2W_1)\,p(R_2W_1) + p(R_3 \mid W_2R_1)\,p(W_2R_1) +$$

$$p(R_3 \mid R_2R_1)\,p(R_2R_1) + p(R_3 \mid W_2W_1)\,p(W_2W_1)$$

$$p(R_3 \mid R_2W_1) = \frac{11}{20}$$

$$p(R_2W_1) = p(R_2 \mid W_1)\,p(W_1) = \frac{10}{22} \cdot \frac{12}{21} = \frac{20}{77}$$

$$\Rightarrow p(R_3) = \frac{12}{22} \quad \#$$

3.4 Bayes' formula

In a bolt factory, 30%, 50%, 20% of production is by machines I, II, III, respectively.

If 4%, 5%, 3% of the output of these respective machines is defective,

What's the prob. that a randomly selected bolt that is found to be defective is by III?

Let A: the event that a random bolt is defective

B_3: the event that it is manufactured by III.

$$P(B_3 \mid A) = \frac{P(B_3 A)}{P(A)}$$

$$\because P(B_3 A) = P(A|B_3)P(B_3)$$

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

$$\therefore P(B_3|A) = \frac{P(A|B_3)P(B_3)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \underbrace{P(A|B_3)}_{0.03}\underbrace{P(B_3)}_{0.2}}$$

$$= \frac{0.03 * 0.2}{0.04 * 0.3 + 0.05 * 0.5 + 0.03 * 0.2} \fallingdotseq 0.14$$

Thm 3.5

$$p(B_k|A) = \frac{p(A|B_k)\,p(B_k)}{p(A|B_1)\,p(B_1) + p(A|B_2)\,p(B_2) + \cdots p(A|B_n)\,p(B_n)}$$

Bayes' Theorem

- 3.5 Independence

$$p(A|B) = p(A)$$

$$\Downarrow$$

$$\frac{p(AB)}{p(B)} = p(A) \qquad \therefore\ p(AB) = p(A)\,p(B)$$

$$\frac{P(BA) = P(AB)}{P(A)} = P(B)$$

$$P(B|A) = P(B)$$

Thm 3.6

If A and B independent, A and B^c are independent

A^c and B^c are independent

Remark 3.3

If A and B are mutually exclusive events, then they are

· dependent

Def.

A, B, C are independent if

$P(AB) = P(A)P(B)$

$P(AC) = P(A)P(C)$

$P(BC) = P(B)P(C)$

$P(ABC) = P(A)P(B)P(C)$

$\underline{P(ABC)} = P(A)\underline{P(B)P(C)}$

$P(BC)$

$P(A(BC)) = P(A)P(BC)$

formally, $P(A(BC)) = P(A)P(BC)$

$P(B(AC)) = P(B)P(AC)$

$P(C(AB)) = P(C)P(AB)$

Ex 3.29 throwing a die twice

Let A be the event that in the second throw the die lands 1, 2, or 5

B in the second throw 4, 5, or 6

C be the event that the sum of the two outcomes is 9.

$P(A) = P(B) = \frac{1}{2}$ $\quad P(C) = \frac{4}{36} = \frac{1}{9}$

$P(AB) = \frac{1}{6} \neq P(A)P(B) = \frac{1}{4}$

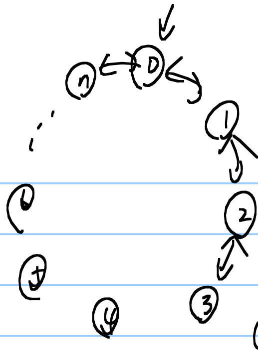
$P(AC) = \frac{1}{36} \neq P(A)P(C)$

$P(BC) = \frac{1}{12} \neq P(B)P(C)$

$P(ABC) = \frac{1}{36} = P(A)P(B)P(C) = \frac{1}{36}$

$$p_i = \frac{a+b-i}{a+b}$$

$$q_i = \frac{a+b-i}{a+b}$$



T: first return

P(all vertices are visited for at least one time)

Conditioning on the first movement

A: a dollars

B: b dollars

fair coin

$$\frac{1}{2} p(\underline{0}) + \frac{1}{2} p(\underline{n})$$

A: initially 1 dollar

B: initially n-1 dollars

$$\frac{1}{2} \cdot \frac{1}{1+(n-1)} + \frac{1}{2} \cdot \frac{1}{1+(n-1)}$$

[illegible] 1

[illegible] n

$$= \frac{1}{n}$$

The set of events $\{A_1, A_2, \dots A_n\}$ is called $\underbrace{\text{independent}}$ if for every subset

$\{A_{i_1}, A_{i_2}, A_{i_3} \dots A_{i_k}\}$ of $\{A_1, A_2, \dots A_n\}$

$$P(A_{i_1}, A_{i_2}, \dots A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

check $\underbrace{2^n - n - 1}$ equations

$$\binom{n}{2} + \binom{n}{3} + \cdots \binom{n}{n} = (1+1)^n - \binom{n}{0} - \binom{n}{1}$$

$$= 2^n - n - 1$$

Ex 3.31

We draw cards, one at a time, at random and successively from an ordinary deck of 52 cards <u>with replacement</u>.

What is the prob. that an ace appears before a face card?

Sol: E: the event of an ace appearing before a face card (j, Q, k)

A, F, B : the events of ace, face, neither in the first experiment

$$p(E) = p(E|A)\,p(A) + p(E|F)\,p(F) + p(E|B)\,p(B)$$

$$= 1 \cdot \frac{4}{52} + 0 \cdot \frac{12}{52} + p(E) \cdot \frac{36}{52}$$

$$p(E) = \frac{4}{52} + \frac{36}{52}\, p(E) \rightarrow p(E) = \frac{1}{4}$$

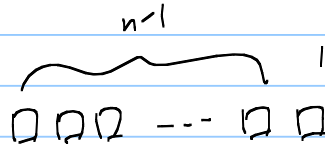
Sol 2:

A_n : the event that no face card or ace appears in the first $(n-1)$ drawings, and the n-th draw is ace.

the event of " an ace before a face card " is $\bigcup_{n=1}^{\infty} A_n$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

∵ mutually exclusive between A_n's



$$P(A_n) = \left(\frac{36}{52}\right)^{n-1} \cdot \frac{4}{52}$$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty}\left(\frac{9}{13}\right)^{n-1}\cdot\frac{1}{13}$$

$$= \frac{1}{13}\sum_{n=1}^{\infty}\left(\frac{9}{13}\right)^{n-1} = \frac{1}{13}\cdot\frac{1}{1-\frac{9}{13}} = \frac{1}{13}\cdot\frac{13}{4}$$

$$= \frac{1}{4}$$

Ex 3.33

Adam tosses a fair coin $n+1$ times, Andrew tosses n times

What is the prob. that Adam gets more heads than Andrew?

H_1: the number of heads obtained by Adam

H_2: " " Andrew

$$P(H_1 > H_2) = P(T_1 > T_2)$$

$$p(T_1 > T_2) = p(n+1-H_1 > n-H_2) = p(H_1 < H_2+1)$$

$$= p(H_1 \leq H_2)$$

$$p(H_1 > H_2) = p(H_1 \leq H_2)$$

since $p(H_1 > H_2) + p(H_1 \leq H_2) = 1$

$$\Rightarrow p(H_1 > H_2) = p(H_1 \leq H_2) = \frac{1}{2}$$

2.

$$p(H_1 > H_2) = \sum_{i=0}^{n} p(H_1 > H_2 \mid H_2 = i)\, p(H_2 = i)$$ ← Law of total probabilities

$$= \sum_{i=0}^{n} p(H_1 > i)\, p(H_2 = i)$$

$$= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} p(H_1 = j)\, p(H_2 = i)$$

where $p(H_1 = j) = \dfrac{C_j^{n+1}}{2^{n+1}}$

$p(H_2 = i) = \dfrac{C_i^{n}}{2^{n}}$

$$\therefore\ p(H_1 > H_2) = \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \frac{\binom{n+1}{j}}{2^{n+1}} \frac{\binom{n}{i}}{2^{n}} = \frac{1}{2^{2n+1}} \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j}\binom{n}{i}$$

$$\sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j} \binom{n}{i} = 2^{n}$$