Ch3: Conditional Probability and Independence

Definition

- *If P(B) > 0, the* ***conditional probability*** *of A given B, denoted by P(A | B), is*

$$P(A|B) =$$

Example 3.2

- From the set of all families with two children, *a family* is selected at random and is found to have a girl.
- What is the probability that the other child of the family is a girl? Assume that in a two-child family all sex distributions are equally probable.

1. Let *B* and *A* be the events that the family has a girl and the family has two girls, respectively.

2. Hence, $P(A \mid B) = \dfrac{P(AB)}{P(B)} =$

3.2 Law of multiplication

$$P(A|B)=\frac{P(AB)}{P(B)}$$

$$\Rightarrow P(AB)=$$

$$P(AB)=P(BA)=$$

Example 3.9

- Suppose that five good fuses and two defective ones have been mixed up.
- To find the defective fuses, we test them one-by-one, at random and without replacement.
- What is the probability that we are lucky and find both of the defective fuses in the first two tests?

1. Di: find a defective fuse in the i-th test
2. $$P(D_1 D_2) = \qquad = \frac{2}{7} \times \frac{1}{6} = \frac{1}{21}$$

- P(ABC) = P(A)P (B | A)P(C | AB)
- **Theorem 3.2** *If* $P(A_1A_2A_3...A_{n-1}) > 0$ *,then*

$$P(A_1A_2A_3...A_{n-1}A_n)$$

$$=$$

3.3 Law of total probability

- **Theorem 3.3 (Law of Total Probability)** Let B be an event with $P(B) > 0$ and $P(B^c) > 0$. Then for any event A,

$$P(A) = P(A \mid B)\, P(B) +$$

Example 3.14 (Gambler's Ruin Problem)

- Two gamblers play the game of "heads or tails," in which each time a fair coin lands heads up player A wins $1 from B, and each time it lands tails up, player B wins $1 from A.
- Suppose that player A initially has *a* dollars and player B has *b* dollars. If they continue to play this game successively, what is the probability that (a) A will be ruined; (b) the game goes forever with nobody winning?

Example 3.14 (Gambler's Ruin Problem)

(a)

1. Let E be the event that A will be ruined if he or she starts with i dollars, and let $p_i = P(E)$.
2. We define F to be the event that A wins the first game

   $\Rightarrow P(E) =$
3. $P(E \mid F) = p_{i+1}$ and $P(E \mid F^c) = p_{i-1}$.

4. $p_i = \frac{1}{2} p_{i+1} + \frac{1}{2} p_{i-1}$,note that : $p_0 = \quad , p_{a+b} =$

5. $p_{i+1} - p_i = p_i - p_{i-1}$

6. Let $p_1 - p_0 = \alpha$

$$\therefore p_{i+1} - p_i = p_i - p_{i-1} = \ldots = p_1 - p_0 = \alpha$$

$$p_1 = p_0 + \alpha$$

$$p_2 = p_1 + \alpha = p_0 + \alpha + \alpha = p_0 + 2\alpha$$

...

$$p_i = p_0 + i\alpha = 1 + i\alpha$$

7. $\because p_{a+b} = 0$

$\therefore$ $\Rightarrow \alpha = \dfrac{-1}{a+b}$

$\Rightarrow p_i =$

(b)

1. The same method can be used with obvious modifications to calculate q_i that B will be ruined if he or she starts with *i* dollars
2. $q_i =$

3. Thus the probability that the game goes on forever with nobody winning is $1-(q_b+p_a)$.
4. But $1-(q_b+p_a) = 1-a/(a+b)-b/(a+b)$ =
5. Therefore, if this game is played successively, eventually either A is ruined or B is ruined.

Definition

- *Let* $\{B_1, B_2, \ldots, B_n\}$ *be a set of nonempty subsets of the sample space S of an experiment. If the events* $B_1, B_2, \ldots, B_n$ *are mutually exclusive and* $\bigcup_{i=1}^{n} B_i = S$, *the set* $\{B_1, B_2, \ldots, B_n\}$ *is called a*   *of S.*

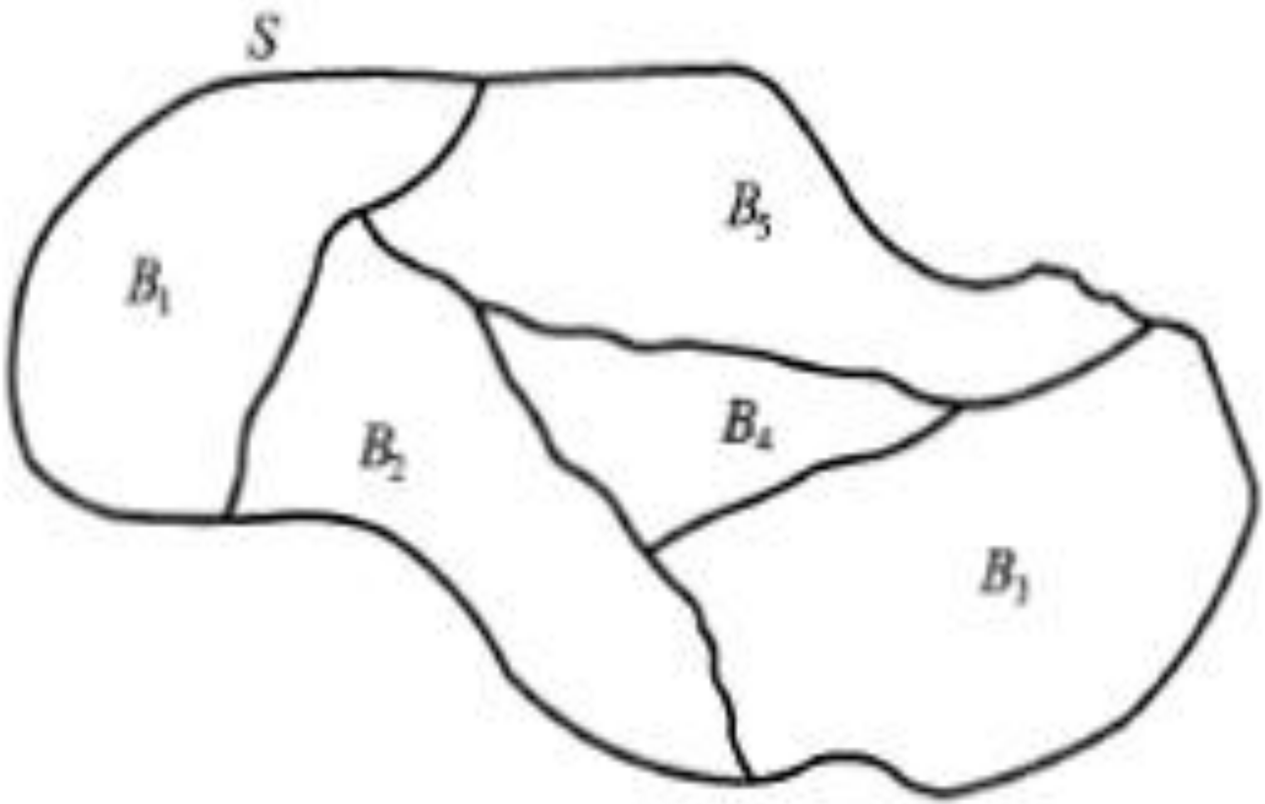

Figure 3.3 Partition of the given sample space S.

Theorem 3.4 (Law of Total Probability)

- *Let* $\{B_1, B_2, \ldots, B_\infty\}$ *be a sequence of mutually exclusive events of S such that* $\bigcup_{i=1}^{\infty} B_i =$

  and suppose that, for all i ≥ 1, P(B_i) > 0. Then for any event A of S,

$$P(A) = \sum_{i=1}^{\infty} P(A \mid B_i) P(B_i)$$

Example 3.17

- An urn contains 10 white and 12 red chips. Two chips are drawn at random and, without looking at their colors, are discarded. What is the probability that a third chip drawn is red?

1. let R_i be the event that the *i*-th chip drawn is red and W_i be the event that it is white.
2. Note that { } is a partition of the sample space
3.

$$P(R_3) = P(R_3 \mid R_2W_1)P(R_2W_1) + P(R_3 \mid W_2R_1)P(W_2R_1)$$
$$+ P(R_3 \mid R_2R_1)P(R_2R_1) + P(R_3 \mid W_2W_1)P(W_2W_1)$$

4.

$$P(R_2W_1) = \qquad\qquad = \frac{12}{21} * \frac{10}{22} = \frac{20}{77}$$

$$P(W_2R_1) = P(W_2 \mid R_1)P(R_1) = \frac{10}{21} * \frac{12}{22} = \frac{20}{77}$$

$$P(R_2R_1) = P(R_2 \mid R_1)P(R_1) = \frac{11}{21} * \frac{12}{22} = \frac{22}{77}$$

$$P(W_2W_1) = P(W_2 \mid W_1)P(W_1) = \frac{9}{21} * \frac{10}{22} = \frac{15}{77}$$

$$\Rightarrow P(R_3) =$$

3.4 Bayes' formula

- In a bolt factory, 30, 50, and 20% of production is manufactured by machines I, II, and III, respectively. If 4, 5, and 3% of the output of these respective machines is defective, what is the probability that a randomly selected bolt that is found to be defective is manufactured by machine III?

1. let A be the event that a random bolt is defective and B_3 be the event that it is manufactured by machine III.

2. $P(B_3 \mid A) =$

$\because P(B_3 A) = P(A \mid B_3) P(B_3),$

$P(A) =$

$$\therefore P(B_3 \mid A) = \frac{P(A \mid B_3) P(B_3)}{P(A \mid B_1) P(B_1) + P(A \mid B_2) P(B_2) + P(A \mid B_3) P(B_3)}$$

$$= \frac{0.03 * 0.2}{0.04 * 0.3 + 0.05 * 0.5 + 0.03 * 0.2} \approx 0.14$$

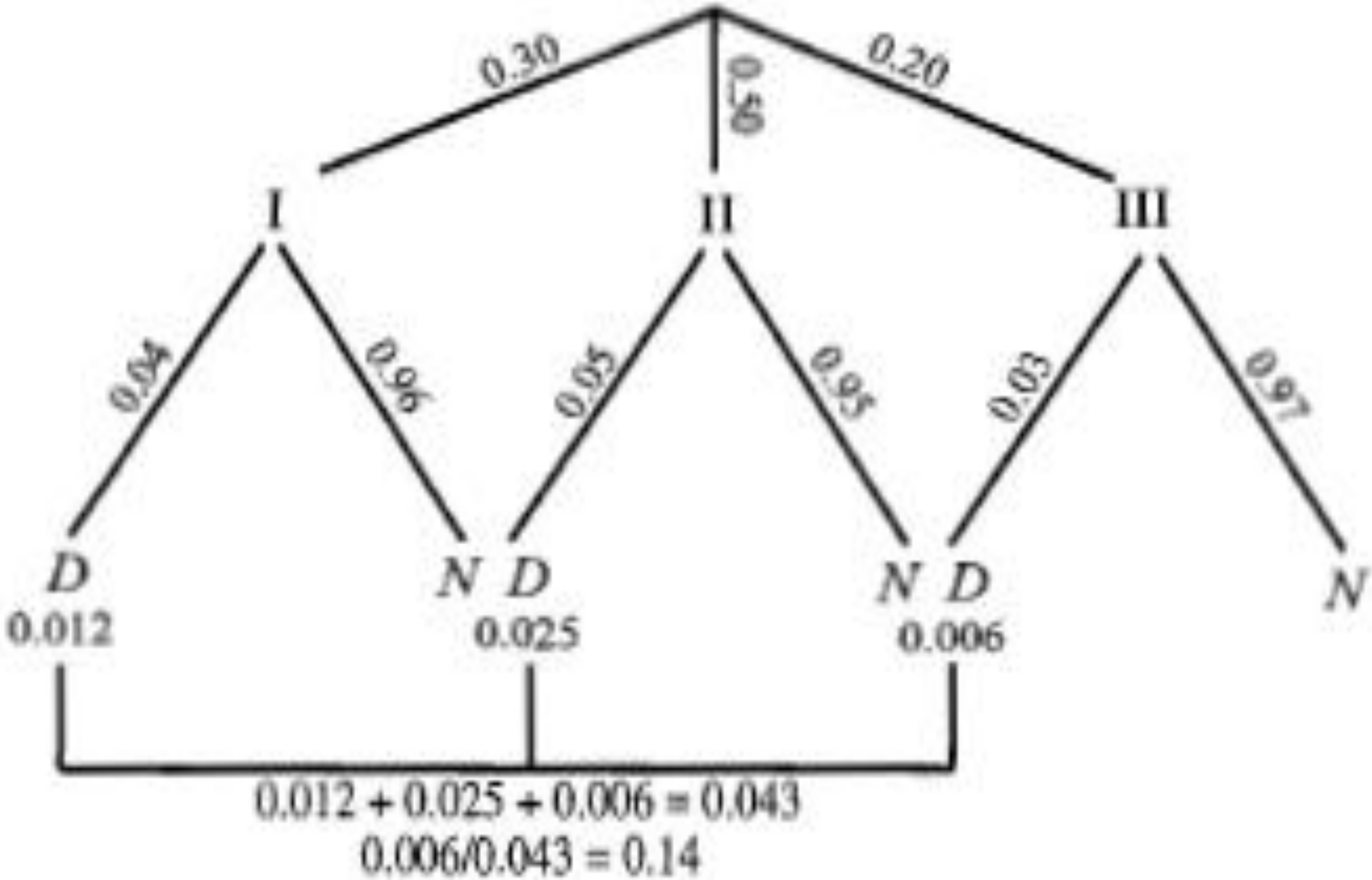

Figure 3.5 Tree diagram for relation (3.14).

Theorem 3.5 (Bayes' Theorem)

- *Let {B1, B2, . . . , Bn} be a partition of the sample space S of an experiment. If for i = 1, 2, . . . , n, P(Bi) > 0, then for any event A of S with P(A) > 0*

$$P(B_k \mid A) = \frac{}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \ldots + P(A \mid B_n)P(B_n)}$$

3.5 Independence

- In general, the conditional probability of A given B is not the probability of A. However, if it is, that is $P(A|B)=P(A)$, we say that A is **independent** of B.

$$P(A|B)=$$
$$P(AB)/P(B)=$$
$$P(AB)=$$

$$P(BA)/P(A)=P(B)$$
$$P(B|A)=P(B)$$

Theorem 3.6

- *If A and B are independent, then A and B^c are as well.*
- **Corollary**
  - *If A and B are independent, then A^c and B^c are as well.*

Remark 3.3

- *If A and B are mutually exclusive events and* $P(A) > 0$, $P(B) > 0$, *then they are*
.
- *This is because, if we are given that one has occurred, the chance of the occurrence of the other one is zero*.

Definition

The events A, B, and C are called ***independent*** *if*

$$P(AB) = P(A)P(B),$$
$$P(AC) = P(A)P(C),$$
$$P(BC) = P(B)P(C),$$
$$P(ABC) =$$

If A, B, and C are independent events, we say that $\{A,B,C\}$ is an independent set of events.

Example 3.29

Let an experiment consist of throwing a die twice. Let *A* be the event that in the second throw the die lands 1, 2, or 5; *B* the event that in the second throw it lands 4, 5 or 6; and *C* the event that the sum of the two outcomes is 9. Then *P(A)* = *P(B)* = 1/2, *P(C)* = 1/9, and

$$P(AB) = \frac{1}{6} \neq \frac{1}{4} = P(A)P(B)$$

$$P(AC) = \frac{1}{36} \neq \frac{1}{18} = P(A)P(C)$$

$$P(BC) = \frac{1}{12} \neq \frac{1}{18} = P(B)P(C)$$

while

$$P(ABC) =$$

Thus the validity of *P(ABC)* = *P(A)P (B)P (C)* is not sufficient for the independence of *A*, *B*, and *C*.

Definition

- The set of events $\{A_1, A_2, ..., A_n\}$ is called **independent** if for every subset

$\{A_{i_1}, A_{i_2}, ..., A_{i_k}\}$,k ≥ 2, of $\{A_1, A_2, ..., A_n\}$

$$P(A_{i_1} A_{i_2} ... A_{i_k}\} =$$

- The number of these equations is $2^n - n - 1$.

Example 3.31

- We draw cards, one at a time, at random and successively from an ordinary deck of 52 cards with replacement. What is the probability that an ace appears before a face card?

Solution 1:

1. E : the event of an ace appearing before a face card.

   A, F, and B : the events of ace, face card, and neither in the first experiment, respectively

2. $P(E) =$

3.

$$P(E) = 1 \times \frac{4}{52} + 0 \times \frac{12}{52} + P(E \mid B) \times \frac{36}{52}$$

$$\because P(E \mid B) = P(E)$$

$$\therefore P(E) = \qquad \Rightarrow P(E) = \frac{1}{4}$$

Solution 2:

1. Let *An* be the event that no face card or ace appears on the first *(n*−1*)* drawings, and the *n*th draw is an ace
2. the event of "an ace before a face card" is $\bigcup_{n=1}^{\infty} A_n$
3. Because of mutually exclusive, $P(\bigcup_{n=1}^{\infty} A_n) =$
4. $P(A_n) =$
5. $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} (\frac{9}{13})^{n-1} (\frac{1}{13})$

$$= (\frac{1}{13}) \sum_{n=1}^{\infty} (\frac{9}{13})^{n-1} = \frac{1}{13} \times \frac{1}{1 - 9/13} = \frac{1}{4}$$

(Geometric series)

Example 3.33

- Adam tosses a fair coin $n + 1$ times, Andrew tosses the same coin n times. What is the probability that Adam gets more heads than Andrew?

Example 3.33

1. Let H_1 and H_2 be the number of heads obtained by Adam and Andrew, respectively. Also, let T_1 and T_2 be the number of tails obtained by Adam and Andrew, respectively. Since the coin is fair,

$$P(H_1 > H_2) = P(T_1 > T_2)$$

But

$$P(T_1 > T_2) = P(n+1-H_1 > n-H_2) = P(H_1 \le H_2)$$

Therefore, .So

$$P(H_1 > H_2) + P(H_1 \le H_2) =$$

implies that $P(H_1 > H_2) = P(H_1 \le H_2) =$

Example 3.33

2. Note that a combinational solution to this problem is neither elegant nor easy to handle:

$$P(H_1 > H_2) =$$

$$= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} P(H_1 = j)P(H_2 = i)$$

$$= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \frac{\frac{(n+1)!}{j!(n+1-j)!}}{2^{n+1}} \frac{\frac{n!}{i!(n-i)!}}{2^{n}}$$

$$= \frac{1}{2^{2n+1}} \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j}\binom{n}{i}$$

Example 3.33

3. However, comparing these two solutions, we obtain the following interesting identity.

$$\sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j} \binom{n}{i} = 2^{2n}$$