Ch 1: Axioms of Probability

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1-2 Sample space and events

- Experiment (Tossing a die)
- Outcome
- S : Sample space = { all possible outcomes }
- Event : subset of sample space

Ex: tossing a coin $S = \{H, T\}$

Ex: flipping a coin and tossing a die if T

flipping a coin if H

$S = \{T1, T2, T3, T4, T5, T6, HT, HH\}$

Ex: A bus with a capacity of 34 passengers stops at a station some time between 11:00 ~ 11:40 AM

$S = \{(i, t) : 0 \le i \le 34,\ 11 \le t \le 11\tfrac{2}{3}\}$

event that the bus arrives between 11:20 ~ 11:40 AM with 29 passengers

$$F = \{ (27, t) : 11\tfrac{1}{3} \le t \le 11\tfrac{2}{3} \}$$

- Event E has occurred in an experiments :

  if the outcome of an experiment belongs to E

- Event : set

subset , equality , intersection , union , complement , difference

- certainty : The sample space is a certain event
- Impossibility : The empty set ϕ , which is S^c , is an impossible event.

- mutually exclusive : If the joint occurrence of two events E and F is impossible $EF = \phi$

$\bigcup_{i=1}^{n} E_i$: at least one of the events occurs

$\bigcap_{i=1}^{n} E_i$: all of the events occur

1.3 Axioms of probability

Def :

S: sample space

A: an event

p : a real-valued function for each event A

$$p : 2^S \to \mathbb{R}$$

2^S ↓ power set : the set of all subsets of S

If p satisfies the following axioms, then it is called a "probability" and the number $p(A)$ is the "prob. of A".

Axiom 1 : $p(A) \geq 0$

Axiom 2 : $p(S) = 1$

Axiom 3 : If $\{A_1, A_2, \cdots\}$ is a sequence of mutually exclusive events, then

$$p\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} p(A_i)$$

Thm 1.1

$$p(\phi) = 0$$

Thm 1.2

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

$\{A_1, A_2, \dots A_n\}$ be a mutually exclusive set of events

Thm 1.3

If S has N points that are all equally likely to occur,

$$P(A) = \frac{N(A)}{N}$$

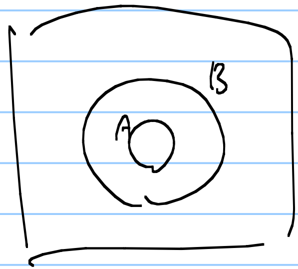
1.4 Basic Theorems

Thm 1.4

$$p(A^c) = 1 - p(A)$$

Thm 1.5

If $A \subseteq B$, then $p(B - A) = p(BA^c) = p(B) - p(A)$



Corollary If $A \subseteq B$, then $p(A) \le p(B)$

Inclusion-Exclusion Principle

Thm 1.6

$$p(A \cup B) = p(A) + p(B) - p(AB)$$

$$p\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} p(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p(A_i A_j) + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} p(A_i A_j A_k)$$

$$- \cdots (-1)^{n-1} p(A_1 A_2 A_3 \cdots A_n)$$

Thm 1.7 $\quad p(A) = p(AB) + p(AB^c)$

1.5 Continuity of Probability function

$f: R \to R$ is called continuous at a point $c \in R$

if $\lim_{x \to c} f(x) = f(c)$

$f: R \to R$ is continuous on R iff

for every convergent sequence $\{x_n\}_{n=1}^{\infty}$ in R

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right)$$

Increasing

- A sequence $\{E_n, n \geq 1\}$ of events of a sample space is

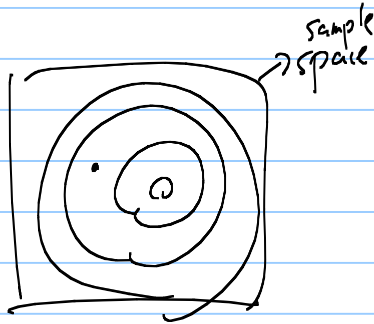
  called "increasing"

if $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \subseteq E_n \subseteq E_{n+1} \cdots$

- For an increasing sequence of events $\{E_n, n \geq 1\}$

$\lim_{n \to \infty} E_n$: the event that "at least" one E_i occurs

$$\lim_{n \to \infty} E_n = \bigcup_{n=1}^{\infty} E_n$$



Decreasing

$E_1 \supseteq E_2 \supseteq E_3 \cdots \supseteq E_n \supseteq E_{n+1} \supseteq \cdots$

$\lim_{n\to\infty} E_n$: the event that every E_n occurs

$$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$$

Tossing a die $\quad E_1 : \{2, 3\}$

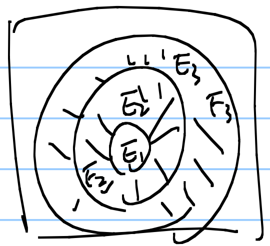
$E_2 : \{2, 3, 4\}$

Thm 1.8 For any increasing or decreasing sequence of events

$$\lim_{n\to\infty} p(E_n) = p(\lim_{n\to\infty} E_n)$$

✓ increasing case

1. Let $F_1 = E_1$, $F_2 = E_2 - E_1$, $\cdots$ $F_n = E_n - E_{n-1}$



$\{F_i, i \geq 1\}$ is a mutually exclusive set of events that satisfies

(1) $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i = E_n$, $n = 1, 2, \cdots$

(2) $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$

$\because$ $\{E_n, n \geq 1\}$ is increasing

$$p(\lim_{n\to\infty} E_n) = p(\bigcup_{i=1}^{\infty} E_i) = p(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} p(F_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} p(F_i)$$

$$= \lim_{n\to\infty} p(\bigcup_{i=1}^{n} F_i) = \lim_{n\to\infty} p(\bigcup_{i=1}^{n} E_i) = \lim_{n\to\infty} p(E_n)$$

- If $\{E_n, n \geq 1\}$ is decreasing $\quad \{E_n^c\}$ is increasing

$$p(\lim_{n\to\infty} E_n) = p(\bigcap_{i=1}^{\infty} E_i) = 1 - p\left(\left[\bigcap_{i=1}^{\infty} E_i\right]^c\right)$$

$$= 1 - P\left(\bigcup_{i=1}^{\infty} E_i^c\right) = 1 - P\left(\lim_{n\to\infty} E_n^c\right)$$

$$= 1 - \lim_{n\to\infty} P(E_n^c)$$

$$= 1 - \lim_{n\to\infty}\left[1 - P(E_n)\right]$$

$$= 1 - 1 + \lim_{n\to\infty} P(E_n) = \lim_{n\to\infty} P(E_n)$$

Ex 1-20.

Suppose that some individuals in a population produce offsprings of the same kind

If with prob. $\exp\left[-\frac{(2n^2+n)}{6n^2}\right]$ the entire population completely dies out by the n-th generation before producing any offspring.

What is the prob. that such a population survives forever?

Ans: Let E_n denote the event of extinction of the entire population by the n-th generation.

$$E_1 \subseteq E_2 \subseteq E_3 \cdots E_n \subseteq E_{n+1} \subseteq \cdots$$

because if E_n occurs then E_{n+1} occurs

By Thm 1.8

$$P[\text{survives forever}] = 1 - P[\text{eventually dies out}]$$

$$= 1 - P\left(\bigcup_{i=1}^{\infty} E_i\right) \longrightarrow \lim_{n\to\infty} E_n$$

$$= 1 - \lim_{n\to\infty} P(E_n)$$

$$\exp\left[\frac{-(2n^2+n)}{6n^2}\right]$$

$$= 1 - \lim_{n\to\infty} \exp\left[\frac{-(2n^2+n)}{6n^2}\right]$$

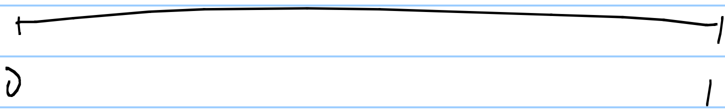
$$= 1 - \exp\left(\frac{-1}{3}\right)$$

1.6 Prob. 0 and 1

Prob. 0

Prob. 1

$$P\left(\frac{1}{3}\right)$$



Ex: $P\left(\frac{1}{3} \text{ is selected}\right)$

0. 529387 - ...

compute the prob. of choosing 0.3333 ...

1. A_n : event that the selected decimal has 3 as its first n digits

$$A_1 \supset A_2 \supset A_3 \supset A_4 \;---$$ decreasing sequence of events

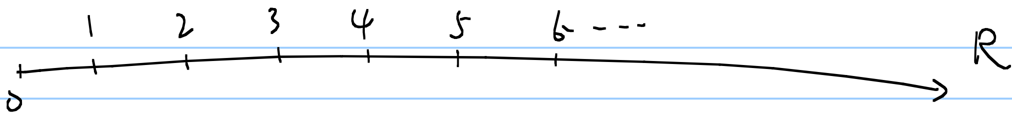
$$p(A_n) = \left(\frac{1}{10}\right)^n$$

$$p\left(\lim_{n\to\infty} A_n\right) = p\left(\bigcap_{n\geq 1}^{\infty} A_n\right)$$

$$p\left(\tfrac{1}{3} \text{ is selected}\right) = \lim_{n\to\infty} p(A_n) = \lim_{n\to\infty} \left(\frac{1}{10}\right)^n = 0$$

It is not correct to say: E is the sample space is $p(E) = 1$ (X)

F is the empty set $\emptyset$ if $p(F) = 0$ ($\times$)



$$p(n \in N) = 0$$

set of natural numbers

1.7 Random selection of points from intervals

The prob. that a random point selected from (a, b)

falls into the interval $\left(a, \frac{a+b}{2}\right)$ is $\frac{1}{2}$

$\left[\frac{a+b}{2}, b\right)$ is $\frac{1}{2}$

$p1$: prob. that point belongs to $(a, \frac{a+b}{2})$

$p2$: ... $[\frac{a+b}{2}, b)$

$$(a, \frac{a+b}{2}) \cup [\frac{a+b}{2}, b) = (a, b)$$

$$p[(a, \frac{a+b}{2}) \cup [\frac{a+b}{2}, b)] = p((a,b)) = 1$$

$$p1 = p2 \qquad \therefore p1 = p_2 = \frac{1}{2}$$

Def: A point is said to be "randomly selected from an interval (a,b)" if any two subintervals of (a, b) that have the same length are equally likely to include the point.

- The prob. of the event that the subinterval (α, β) contains the point is

$$\frac{\beta - \alpha}{b - a}$$