Ch 11 - Part I : Sums of Independent R.V.s

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Moment-generating functions

<u>domain transform</u>

$$M_X(t) = E[e^{tX}]$$

discrete

$$M_X(t) = \sum_{x \in A} e^{tx} p(x)$$

continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

Thm 11.1

$$E[X^n] = M_X^{(n)}(0)$$

($M_X^{(n)}(t)$ is the nth derivative of $M_X(t)$)

Proof :

$$M_X'(t) = \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} x e^{tx} f(x)\,dx$$

$$M_X''(t) = \frac{d}{dt}\int_{-\infty}^{\infty} x e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} x^2 e^{tx} f(x)\,dx$$

$$\vdots$$

$$M_X^{(n)}(t) = \int_{-\infty}^{\infty} x^n e^{tx} f(x)\,dx$$

$$M_X^{(n)}(0) = \int_{-\infty}^{\infty} x^n f(x)\,dx = E[X^n]$$

Ex 11-1

X: Bernoulli r.v. with p

$$p(X=x) = \begin{cases} 1-p & \text{if } x=0 \\ p & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

$M_X(t)$ and $E[X^n]$

$$M_X(t) = E[e^{tX}] = e^{t\cdot 0}\cdot(1-p) + e^{t\cdot 1}\cdot p = 1-p+pe^t$$

$$E(X^n] = M_X^{(n)}(t)\Big|_{t=0} = pe^t\Big|_{t=0} = p$$

Ex 11.2

$X \sim$ Binomial (n, p)

$\Rightarrow M_X(t)$, $E(X]$, $Var(X]$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \leftarrow \text{binomial mass}$$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x \underbrace{(1-p)}_{q}{}^{n-x} = (pe^t + q)^n$$

$$M_X'(t) = npe^t(pe^t+q)^{n-1}$$

$$M_X''(t) = npe^t(pe^t+q)^{n-1} + n(n-1)(pe^t)^2(pe^t+q)^{n-2}$$

$$E[X] = M_X'(0) = np$$

$$E[X^2] = M_X''(0) = np + n(n-1)p^2$$

$$Var(X) = E[X^2] - (E[X])^2 = npq$$

Thm 11.2

Let X and Y be two r.v.s

with m.g.f.s $M_X(t)$ and $M_Y(t)$.

If $M_X(t) = M_Y(t)$, then X and Y have the same distribution.

m.g.f $\Leftrightarrow$ distribution function uniqueness

Ex 1.1.7

$$M_X(t) = \frac{1}{7}e^{t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{7t}$$

another r.v. with p.m.f.

i	1	3	5	7	Other values
$p(i)$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0

m.g.f. $E[e^{tX}] = \sum_{\text{all } i} e^{ti} p(X=i) = e^{t} \cdot \frac{1}{7} + e^{3t} \cdot \frac{3}{7} + e^{5t} \cdot \frac{2}{7} + e^{7t} \cdot \frac{1}{7}$

Theorem 11.2 shows that if two r.v.s have the same m.g.f., then they are identically distributed.

Ex 11-8

Let X be a r.v. with m.g.f. $M_X(t) = e^{2t^2}$.

Find $P(0<X<1)$.

Sol: $M_X(t) = e^{2t^2}$ $\shortparallel$

Normal m.g.f : $\exp[\mu t + \frac{1}{2}\sigma^2 t^2]$

$\mu \to 0$, $\sigma^2 \to 4$

$X \sim N(\mu = 0, \sigma^2 = 4)$

$Z = \frac{X-0}{2}$ ~ standard normal

$p(0 < X < 1) = p(0 < \frac{X}{2} < \frac{1}{2}) = p(0 < Z < \frac{1}{2})$

$= \Phi(0.5) - \Phi(0) \approx 0.6915 - 0.5$

$= 0.1915$

Generating function → Distribution (mass. density) function

$$M_X(t) = E[e^{tX}] = E[z^X]$$

Partial fraction expansion

Type I :

$$\frac{a+bz}{(1-cz)(1-dz)} = \frac{A}{(1-cz)} + \frac{B}{(1-dz)}$$

Type II :

$$\frac{a+bz+cz^2}{(1-dz)^2(1-ez)} = \frac{A}{(1-dz)^2} + \frac{B}{(1-dz)} + \frac{C}{(1-ez)}$$

An example

$$M_X(t) = \frac{4}{(2-z)(3-z)^2}$$

" $Pr[X=n]$ "

$\Rightarrow \{p_n\}$

$\underbrace{\hphantom{p_n}}$ probability distribution ?

$$\frac{4}{(2-z)(3-z)^2} = \frac{A}{(2-z)} + \frac{B}{(3-z)} + \frac{C}{(3-z)^2}$$

$$A(3-z)^2 + B(2-z)(3-z) + C(2-z) = 4$$

$z \leftarrow 2$

$A \cdot (3-2)^2 = 4$

$A = 4$

$z \leftarrow 3$ $\quad C \cdot (2-3) = 4$

$C = -4$

$z \leftarrow 0$

$4 \cdot 9 + B(2)(3) + (-4) \cdot 2 = 4$

$B = -4$

$$M_X(t) = \frac{4}{(2-z)} - \frac{4}{(3-z)} - \frac{4}{(3-z)^2}$$

$$= \frac{2}{(1-\frac{1}{2}z)} - \frac{\frac{4}{3}}{(1-\frac{1}{3}z)} - \frac{\frac{4}{9}}{(1-\frac{1}{3}z)^2}$$

$$p_n = Pr[X=n] = \underline{2\cdot\left(\frac{1}{2}\right)^n} - \underline{\frac{4}{3}\left(\frac{1}{3}\right)^n - \frac{4}{9}\left(\frac{1}{3}\right)^n(n+1)}$$

$$p_n = \left(\frac{1}{2}\right)^n \quad n=0,1,2,\cdots$$

$$E[z^X] = \sum_{i=0}^{\infty} z^i p(X=i) = \sum_{i=0}^{\infty} z^i \left(\frac{1}{2}\right)^i = \sum_{i=0}^{\infty}\left(\frac{1}{2}z\right)^i = \frac{1}{\left(1-\frac{1}{2}z\right)}$$

$$\sum_{n=0}^{\infty} \underbrace{\frac{4}{9}\left(\frac{1}{3}\right)^n(n+1)}_{\text{mass function}} z^n = \left[\sum_{n=0}^{\infty}\frac{4}{3}\left(\frac{1}{3}z\right)^n\right]' = \left[\frac{4}{3}\frac{1}{1-\frac{1}{3}z}\right]'$$

$$= \frac{-\frac{4}{3}\cdot -\frac{1}{3}}{\left(1-\frac{1}{3}z\right)^2} = \frac{\frac{4}{9}}{\left(1-\frac{1}{3}z\right)^2}$$

1

2

② $\frac{4}{3}\cdot\left(\frac{1}{3}z\right)^0 + \frac{4}{3}\left(\frac{1}{3}z\right)^1 + \frac{4}{3}\left(\frac{1}{3}z\right)^2$

$$\frac{4}{3}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}z\right)^0 + \frac{4}{3}\cdot 2\cdot\left(\frac{1}{3}z\right)^1\cdot\frac{1}{3} + \cdots$$

① $\frac{4}{9}\cdot\left(\frac{1}{3}z\right)^0\cdot 1 + \frac{4}{9}\left(\frac{1}{3}z\right)^1\cdot 2$

11.2 Sum of independent r.v.s

Thm 11.3

$X_1, X_2 \cdots X_n$ independent

$\downarrow \quad \downarrow \quad \downarrow$

$M_{X_1}(t) \quad M_{X_2}(t) \quad M_{X_n}(t)$

$x_1 + x_2 + \cdots x_n$

$$M_{x_1+x_2+\cdots x_n}(t) = M_{x_1}(t)\, M_{x_2}(t) \cdots M_{x_n}(t)$$

Proof: $W = x_1 + x_2 + \cdots x_n$

$$M_w(t) = E[e^{tW}] = E[e^{tx_1 + tx_2 + \cdots tx_n}]$$

$$= E[e^{tx_1} e^{tx_2} \cdots e^{tx_n}]$$

$$= E[e^{tx_1}]\, E[e^{tx_2}] \cdots E[e^{tx_n}]$$

$$= M_{x_1}(t)\, M_{x_2}(t) \cdots M_{x_n}(t)$$

Thm 8.6

Let X and Y be independent r.v.s

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

Sol:

$$E[g(X)h(Y)] = \sum_{x \in A}\sum_{y \in B} g(x)h(y)\underbrace{p(x,y)}_{\text{independent}}$$

$$= p_X(x)\,p_Y(y)$$

$$= \sum_{x \in A}\sum_{y \in B} g(x)h(y)\,p_X(x)\,p_Y(y)$$

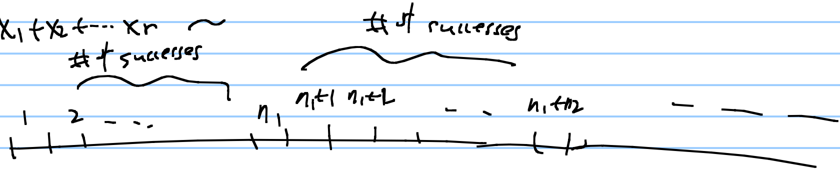
$$= \underbrace{\sum_{x \in A} g(x) p_X(x)}\, E[h(Y)]$$

$$= E[g(X)]\, E[h(Y)]$$

Thm 11.4

$X_1, X_2 \cdots X_r$ independent binomial r.v.s with $(n_1, p), (n_2, p) \cdots (n_r, p)$

$\Rightarrow X_1 + X_2 + \cdots X_r \sim$ Binomial$(n_1 + n_2 + \cdots n_r, p)$



Proof :

$$M_{X_i}(t) = (pe^t + q)^{n_i}$$

$$W = X_1 + X_2 + \cdots X_r$$

$$M_W(t) = M_{X_1}(t)\, M_{X_2}(t)\, M_{X_3}(t) \cdots M_{X_r}(t)$$

$$= (pe^t + q)^{n_1} (pe^t + q)^{n_2} \cdots (pe^t + q)^{n_r}$$

$$= (pe^t + q)^{n_1 + n_2 + \cdots n_r}$$

By uniqueness property

$W \sim$ binomial $(n_1 + n_2 + \cdots n_r,\ p)$

Thm 11.5 $X_1 \; X_2 \; \cdots \; X_n$ independent poisson $\lambda_1, \lambda_2, \cdots \lambda_n$

$W = X_1 + \cdots X_n \sim$ poisson $(\lambda_1 + \lambda_2 + \cdots \lambda_n)$

Ans:

$$M_Y(t) = E[e^{tY}] = \sum_{y=0}^{\infty} e^{ty} \cdot \frac{e^{-\lambda}\lambda^y}{y!} = e^{-\lambda} \underbrace{\sum_{y=0}^{\infty} \frac{(e^t\lambda)^y}{y!}}_{e^{e^t\lambda}} = e^{-\lambda} \cdot e^{e^t\lambda}$$

$Y \sim$ poisson (λ)

$$= \exp[\lambda(e^t - 1)]$$

Let $W = X_1 + \cdots X_n$

$$M_W(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t) = \exp[\lambda_1(e^t - 1)] \exp[\lambda_2(e^t - 1)] \cdots \exp[\lambda_n(e^t - 1)]$$

$$= \exp[(\lambda_1 + \lambda_2 + \cdots \lambda_n)(e^t - 1)]$$

By uniqueness property

$\Rightarrow W \sim \text{Poisson}(\underbrace{\lambda_1 + \lambda_2 + \cdots \lambda_n}_{\text{mean}})$

Thm 11.6

$X_1 \sim N(\mu_1, \sigma_1^2) \quad X_2 \sim N(\mu_2, \sigma_2^2) \cdots \quad X_n \sim N(\mu_n, \sigma_n^2)$

independent r.v.s

$X_1 + X_2 + \cdots X_n \sim N(\mu_1 + \mu_2 + \cdots \mu_n, \ \sigma_1^2 + \sigma_2^2 + \cdots \sigma_n^2)$

Proof:

$M_X(t) = \exp[\mu t + \frac{1}{2}\sigma^2 t^2]$

$W = X_1 + \cdots X_n$

$M_W(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$

$= \exp[\mu_1 t + \frac{1}{2}\sigma_1^2 t^2]\, \exp[\mu_2 t + \frac{1}{2}\sigma_2^2 t^2] \cdots \exp[\mu_n t + \frac{1}{2}\sigma_n^2 t^2]$

$= \exp[(\mu_1 + \mu_2 + \cdots \mu_n)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \cdots \sigma_n^2)t^2]$

Other similar theorems

- Sums of independent geometric r.v.s are Negative Binomial
- Sums of independent negative binomial are Negative binomial

- Sums of independent exponential r.v.s are gamma
- Sums of independent gamma r.v.s are gamma

If $X_1, X_2 \dots X_n$ independent gamma

(r_1, λ) (r_2, λ) (r_n, λ)

$X_1 + \dots X_n \sim \text{gamma}(r_1 + \dots r_n, \lambda)$

Ex 11.10

- office fire insurance policies have a \$ 1000 deductible
- received three claims, independent of each other
- reconstruction expenses for such claims are

exponential distributed with mean \$ 45,000.

$\Rightarrow$ prob. that the total payment for these claims is < \$120,000 ?

Ans: Let X be the total expenses

$X \sim \text{gamma}\left(r=3,\ \lambda=\frac{1}{45}\right)$

$$f(x)=\begin{cases}\frac{1}{45}e^{\frac{-x}{45}}\,\frac{\left(\frac{x}{45}\right)^2}{2}=\frac{1}{182250}x^2e^{\frac{-x}{45}} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

$$p(X<123)=\frac{1}{182250}\int_0^{123}x^2e^{\frac{-x}{45}}\,dx=\frac{1}{182250}\left(-45x^2-4050x-182250\right)e^{\frac{-x}{45}}\Big|_0^{123}$$

$= 0.5145$