Ch11 - Part I: Sums of Independent Random Variables

11.1 Moment-generating functions

- **Definition** *For a random variable X, let*

$$M_X(t) = E(e^{tX}).$$

$M_X(t)$ is called the
function *of X.*

- If *X* is a discrete random variable with set of possible values *A* and probability mass function *p(x)*, then

$$M_X(t) = \sum_{x \in A} e^{tx} p(x)$$

- If *X* is a continuous random variable with probability density function *f(x)*, then

$$M_X(t) =$$

Theorem 11.1

- *Let X be a random variable with moment-generating function $M_X(t)$. Then*

$$E(X^n) =$$

where $M_X^{(n)}(t)$ is the nth derivative of $M_X(t)$.

proof :

1. Let X be continuous with p.d.f f.

$$M'_X(t) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} e^{tx} f(x)dx\right) = \int_{-\infty}^{\infty} xe^{tx} f(x)dx$$

$$M''_X(t) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} xe^{tx} f(x)dx\right) = \int_{-\infty}^{\infty} x^2 e^{tx} f(x)dx$$

$$\vdots$$

$$M_X^{(n)}(t) =$$

where we have assumed that the derivatives of these integrals are equal to the integrals of the derivatives of their integrands.

2. Letting t = 0, we get

$$M_X^{(n)}(0) = \int_{-\infty}^{\infty} x^n f(x)dx =$$

Example 11.1

- Let X be a Bernoulli random variable with parameter p, that is,

$$P(X = x) = \begin{cases} 1-p & \text{if x} = 0 \\ p & \text{if x} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine $M_X(t)$ and $E(X^n)$.

Solution: From the definition of a moment-generating function,

$$M_X(t) = E(e^{tX}) =$$

Since $M_X^{(n)}(t) = pe^t$ for all $\mathrm{n} > 0$, we have that $E(X^n) =$

Example 11.2

- Let X be a binomial random variable with parameters (n, p).
- Find the moment-generating function of X, and use it to calculate $E(X)$ and $\mathrm{Var}(X)$.

Example 11.2

Solution:

1. The p.m.f of X, p(x), is given by

$$p(x) = \binom{n}{x} p^x q^{n-x}, \qquad x = 0,1,2,\ldots, n, \quad q = 1 - p.$$

2. Hence, $M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (pe^t + q)^n \, (Theorem\, 2.5)$

3. To find the mean and the variance of X, note that

$M'_X(t) = npe^t (pe^t + q)^{n-1}$

$M''_X(t) =$

Thus

$E(X) = M'_X(0) = np$

$E(X^2) = M''_X(0) = np + n(n-1)p^2$

Therefore, $\quad Var(X) = E(X^2) - [E(X)]^2$

$=$

Example 11.3

- Let X be an exponential random variable with parameter λ.
- Using moment-generating functions, calculate the mean and the variance of X.

Example 11.3

Solution:

1. The p.d.f of X is given by $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$
2. $M_X(t) = E(e^{tX}) =$
3. Since the integral $\int_0^\infty e^{(t-\lambda)x}dx$ converges if $t < \lambda$, restricting the domain of $M_X(t)$ to $(-\infty, \lambda)$, we get $M_X(t) = \lambda/(\lambda - t)$.
4. $M'_X(t) = \lambda/(\lambda - t)^2$ and $M''_X(t) = (2\lambda)/(\lambda - t)^3$

$E(X) = M'_X(0) = 1/\lambda$

$E(X^2) = M''_X(0) = 2/\lambda^2$

Therefore, $\quad Var(X) = E(X^2) - [E(X)]^2 =$

Theorem 11.2

- *Let X and Y be two random variables with moment-generating functions $M_X(t)$ and $M_Y(t)$. If $M_X(t) = M_Y(t)$, then X and Y have the*
- Theorem 11.2 shows that if two random variables have the same moment-generating function, then they are

Example 11.7

- Let the moment-generating function of a random variable X be

$$M_X(t) = \frac{1}{7}e^{t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{7t}$$

Since the moment-generating function of a discrete random variable with p.m.f

i	1	3	5	7	Other values
$p(i)$	1/7	3/7	2/7	1/7	0

is $M_X(t)$. By theorem 11.2, the p.m.f of X is $p(i)$.

Example 11.8

- Let X be a random variable with moment-generating function $M_x(t)=e^{2t^2}$. Find P(0<X<1).
- Solution:
  - Comparing $M_x(t)=e^{2t^2}$ with $\exp[\mu t +(1/2)\sigma^2 t^2]$, the moment-generating function of $N(\mu, \sigma^2)$, we have that by the uniqueness of the moment-generating function.

Example 11.8

- Let $Z=(X-0)/2$. Then $Z\sim N(0,1)$, so

$$P(0<X<1)=P(0<X/2<1/2)=P(0<Z<0.5)$$

$$=$$

Supplement: From Generating Function to Distribution Function

Consider a discrete random variable X for illustration. Let

$$M_X(t) = E[e^{tX}] = E[z^X]$$

Partial fraction expansion

- Type I
$$\frac{a+bz}{(1-cz)(1-dz)} =$$
where $c \neq d$.

Supplement: From Generating Function to Distribution Function

Partial fraction expansion

- Type I

$$\frac{a+bz}{(1-cz)(1-dz)}=$$

where $c \neq d$.

- Type II: in which the denominator factors are not distinct.

$$\frac{a+bz+cz^2}{(1-dz)^2(1-ez)}=$$

where $d \neq e$.

An Example

- Assume that we are given the following generating function of random variable X

$$M_X(t) = \frac{4}{(2-z)(3-z)^2}.$$

- What is the probability distribution $\{p_n\}$ of X?

An Example

Ans: We need to invert M_X(t). First, we write

$$\frac{4}{(2-z)(3-z)^2} =$$

Multiplying that by the denominator on the left side gives

$$A(3-z)^2 + B(2-z)(3-z) + C(2-z) = 4.$$

Setting z at 2, 3, and 0 in succession, we find

An Example

- The partial fraction expansion of $M_X(t)$ is then given by

$$M_X(t) =$$

- We invert the previous equation to the time domain

$$p_n = 2\left(\frac{1}{2}\right)^n - \left(\frac{4}{3}\right)\left(\frac{1}{3}\right)^n - \left(\frac{4}{9}\right)\left(\frac{1}{3}\right)^n (n+1),$$

where n=0, 1, 2, …

11.2 Sum of independent random variables

Theorem 11.3

- *Let $X_1, X_2, \ldots, X_n$ be independent random variables with moment-generating functions $M_{X1}(t), M_{X2}(t), \ldots, M_{Xn}(t)$.*
- *The moment-generating function of $X_1 + X_2 + \cdots + X_n$ is given by $M_{X1+X2+\cdots+Xn}(t) =$*

Proof:

Let $W = X_1 + X_2 + \ldots + X_n$; by definition,

$$
\begin{aligned}
M_W(t) &= E(e^{tW}) = E(e^{tX_1 + tX_2 + \ldots + tX_n}) \\
&= E(e^{tX_1} e^{tX_2} \ldots e^{tX_n}) \\
&= E(e^{tX_1}) E(e^{tX_2}) \ldots E(e^{tX_n}) \\
&=
\end{aligned}
$$

where the next-to-last equality follows from the independence of $X_1, X_2, \ldots, X_n$.

Theorem 8.6 (see p.333)

- *Let X and Y be independent random variables. Then for all real-valued functions* $g: \mathbf{R} \rightarrow \mathbf{R}$ *and* $h: \mathbf{R} \rightarrow \mathbf{R}$*,*

E[g(X)h(Y)] =

where, as usual, we assume that E[g(X)] and E[h(Y)] are finite.

Theorem 8.6

- Solution: Let A be the set of possible values of *X*, and B be the set of possible values for *Y*. Let *p*(*x,y*) be the joint probability mass function of *X* of *Y*. Then

$$E[g(X)h(Y)] = \sum_{x\in A}\sum_{y\in B} g(x)h(y)p(x,y) = \sum_{x\in A}\sum_{y\in B} g(x)h(y)p_x(x)p_y(y) =$$

$$= \sum_{x\in A}[g(x)p_x(x)\sum_{y\in B} h(y)p_y(y)] = \sum_{x\in A} g(x)p_x(x)E[h(Y)]$$

$$=$$

Theorem 11.4

- *Let* $X_1, X_2, \dots, X_r$ *be independent binomial random variables with parameters* $(n_1, p), (n_2, p), \dots, (n_r, p)$*, respectively.*
- *Then* $X_1 + X_2 + \cdots + X_r$ *is a binomial random variable with parameters*

Theorem 11.4

- Proof: Let, as usual, q=1-p. We know that $M_{xi}(t)=(pe^t+q)^{ni}$, i=1,2,3,…,r
  - Let $W=X_1+X_2+\ldots+X_r$; then, by Theorem 11.3,

$$M_W(t) = M_{x1}(t)M_{x2}(t)\ldots M_{xr}(t)$$

$$= (pe^t+q)^{n1}(pe^t+q)^{n2}\ldots(pe^t+q)^{nr}$$

$$=$$

Theorem 11.4

- Since $(pe^t+q)^{n1+n2+\ldots+nr}$ is the moment-generating function of a binomial random variable with parameters $(n_1+n_2+\ldots+n_r, p)$, the uniqueness property of moment-generating functions implies that $W=X_1+X_2+\ldots+X_r$ is binomial with parameters

Theorem 11.5

- *Let* $X_1, X_2, \ldots, X_n$ *be independent Poisson random variables with means* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *respectively. Then* $X_1 + X_2 + \cdots + X_n$ *is a Poisson random variable with mean*

Theorem 11.5

- Proof: Let *Y* be a Poisson random variable with mean *λ*. Then

$$M_Y(t) = E(e^{tY}) = \sum_{y=0}^{\infty} e^{ty} \frac{e^{-\lambda}\lambda^y}{y!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{(e^t \lambda)^y}{y!}$$

$$=$$

Let $W=X_1+X_2+\ldots+X_n$; then, by Theorem 11.3,

$$M_W(t) = M_{X1}(t) M_{X2}(t) \ldots M_{Xn}(t)$$

$$= \exp[\lambda_1(e^t - 1)] \exp[\lambda_2(e^t - 1)] \ldots \exp[\lambda_n(e^t - 1)]$$

$$=$$

Theorem 11.5

- Now, since $\exp[(\lambda_1+\lambda_2+\ldots+\lambda_n)(e^t-1)]$ is the moment-generating function of a Poisson random variable with mean $\lambda_1+\lambda_2+\ldots+\lambda_n$, the uniqueness property of moment-generating functions implies that $X_1+X_2+\ldots+X_n$ is Poisson with mean

Theorem 11.6

- Let $X_1 \sim N(\mu_1, \sigma_1^2), X_2 \sim N(\mu_2, \sigma_2^2), \ldots, X_n \sim N(\mu_n, \sigma_n^2)$ be independent random variables. Then $X_1 + X_2 + \ldots + X_n \sim$
- Proof:

  In Example 11.5 we showed that if X is normal with parameters μ and σ^2, then $M_X(t) = \exp[\mu t + (1/2)\sigma^2{}_1 t^2]$.

Theorem 11.6

- Let *W=X₁+X₂+…+Xₙ*; then

$$M_W(t) = M_{X1}(t)M_{X2}(t)...M_{Xn}(t)$$

$$= \exp(\mu_1 t + \frac{1}{2}\sigma_1^2 t^2)\exp(\mu_2 t + \frac{1}{2}\sigma_2^2 t^2)...\exp(\mu_n t + \frac{1}{2}\sigma_n^2 t^2)$$

$$=$$

- This implies that

$$X_1 + X_2 + ... + X_n \sim N(\mu_1 + \mu_2 + ... + \mu_n, \sigma_1^2 + \sigma_2^2 + ... + \sigma_n^2)$$

Other similar theorems

- Sums of independent geometric random variables are
- Sums of independent negative binomial random variables are
- Sums of independent exponential random variables are
- Sums of independent gamma random variables are

- Specifically, we can prove, by using moment-generating functions, that
  - if X_1, X_2, …, X_n are n independent gamma random variables with parameters (r_1,λ), (r_2,λ), …, (r_n,λ), respectively, then $X_1 + X_2 + \dots + X_n$ is gamma with parameters

Example 11.10

- Office fire insurance policies by a certain company have a $1000 deductible.
- The company has received three claims, independent of each other, for damages caused by office fire.
- If reconstruction expenses for such claims are exponentially distributed, each with mean $45,000, what is the probability that the total payment for these claims is less than $120,000?

1. Let X be the total reconstruction expenses for the three claims in thousands of dollars;

X is the sum of three independent exponential random variables, each with parameter $45. Therefore, it is a with parameters 3 and *λ = 1/45.*

2.

$$f(x) = \begin{cases} \\ \end{cases}$$

3. $$P(X < 123) = \frac{1}{182250}\int_0^{123} x^2 e^{-x/45}dx$$

$$= \frac{1}{182250}(-45x^2 - 4050x - 182250)e^{-x/45}\Big|_0^{123} = 0.5145$$