Ch10: More Expectations and Variances

Part I: Expected Values of Sums of Random Variables

10.1 Expected values of sums of random variables

- **Theorem 10.1** *For random variables $X_1, X_2, \ldots, X_n$ defined on the same sample space,*

$$E\left(\sum_{i=1}^{n} \alpha_i X_i\right) =$$

- **Corollary** *Let $X_1, X_2, \ldots, X_n$ be random variables on the same sample space. Then*
  $\boldsymbol{E(X_1 + X_2 + \cdots + X_n) =}$

- This is the generalization of the corollaries of Thm 8.1 and Thm 8.2.

Example 10.1

- A die is rolled 15 times. What is the expected value of the sum of the outcomes?

1. Let X be the sum of the outcomes, and for i=1,2,...,15. Let X_i be the outcome of the ith roll. Then $X = X_1 + X_2 + \cdot\cdot\cdot + X_{15}$. Thus
2. $E(X) =$

   $E(X_i) =$

   Hence, $E(X) = 15(7/2) = 52.5$

Example 10.2

- A well-shuffled ordinary deck of 52 cards is divided randomly into four piles of 13 each.
- Counting jack, queen, and king as 11, 12, and 13, respectively, we say that a match occurs in a pile if the jth card is j.
- What is the expected value of the total number of matches in all four piles?

Example 10.2

1. Let Xi, i=1,2,3,4 be the number of matches in the ith pile. X=X1+X2+X3+X4 is the total number of matches.
2.

$$E(X)=$$

To calculate $E(X_i), 1 \le i \le 4$,

Let $\mathrm{A_{ij}}$ be the event that the jth card in the ith pile is j

$(1 \le \mathrm{i} \le 4, 1 \le j \le 13)$. Then by defining

$$X_{ij} =$$

we have that $\mathrm{X_i} = \sum_{j=1}^{13} X_{ij}$. Now $\mathrm{P(A_{ij})} =$

$$\Rightarrow E(X_{ij}) =$$

$$\text{Hence, } \mathrm{E(X_i)} = E\left(\sum_{j=1}^{13} X_{ij}\right) = \sum_{j=1}^{13} \frac{1}{13} = 1$$

$$E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) = 1+1+1+1 = 4$$

Example 10.3

- Exactly *n* married couples are living in a small town.
- What is the expected number of intact couples after *m* deaths occur among the couples?
- Assume that the deaths occur at random, there are no divorces, and there are no new marriages.

Example 10.3

1. Let X be the number of intact couples after m death,

and for i = 1,2,..., n define $X_i = \begin{cases} 1 & \text{if the ith couple is left intact} \\ 0 & \text{otherwise} \end{cases}$

Hence, $E(X) = E(X_1) + E(X_2) + \ldots + E(X_n)$ where

$\mathrm{E}(\mathrm{X}_\mathrm{i}) =$

$P(X_i = 1) =$

$$E(X) = n \cdot P(X_i = 1) = \frac{(2n-m)(2n-m-1)}{2(2n-1)}$$

Example 10.4

- Dr. Windler's secretary accidentally threw a patient's file into the wastebasket.
- A few minutes later, the janitor cleaned the entire clinic, dumped the wastebasket containing the patient's file randomly into one of the seven garbage cans outside the clinic, and left.
- Determine the expected number of cans that Dr. Windler should empty to find the file.

Example 10.4

1. Let X be the number of garbage cans that Dr. Windler should empty to find the patient's file.

2. For $i = 1, 2, \ldots, 7$, let $X_i = 1$ if the patient's file is in the ith garbage can that Dr. Windler will empty, and $X_i = 0$, otherwise.

Then, $X = 1 \cdot X_1 + 2 \cdot X_2 + \ldots + 7 \cdot X_7$

therefore,

$E(X) =$

Example 10.5

- A box contains nine light bulbs, of which two are defective.
- What is the expected value of the number of light bulbs that one will have to test (at random and without replacement) to find both defective bulbs?

Example 10.5

1. For $i = 1,2,\ldots,8$ and $j > i$, let $X_{ij} = j$ if the ith and jth light bulbs to be examined are defective, and $X_{ij} = 0$ otherwise.

Therefore,

$E(X) =$

≈ 6.67

Example 10.6

- Let X be a binomial random variable with parameters (n, p). Recall that X is the number of successes in n independent Bernoulli trials. Thus, for $i = 1, 2, \ldots, n$, letting

$$X_i = \begin{cases} 1 & \text{if the ith trial is a success} \\ 0 & \text{otherwise} \end{cases}$$

we get

$$X = X_1 + X_2 + \cdots + X_n, \qquad (10.1)$$

where X_i is a Bernoulli random variable for $i = 1, 2, \ldots, n$.

Example 10.6

- Now, since $\forall i$, $1 \leq i \leq n$,

  $E(X_i) =$

  (10.1) implies that

  $$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = np$$

Example 10.7

- Let X be a negative binomial random variable with parameters (r, p).
- Then in a sequence of independent Bernoulli trials each with success probability p, X is the number of trials until the rth success.
- Let X_1 be the number of trials until the first success, X_2 be the number of additional trials to get the second success, X_3 the number of additional ones to obtain the third success, and so on.

Example 10.7

- Then clearly
$$X = X_1 + X_2 + \cdot\cdot\cdot + X_r,$$
where for $i = 1, 2, \ldots, n$, the random variable X_i is geometric with parameter p.
- This is because $P(X_i = n) = (1-p)^{n-1}p$ by the independence of the trials.
- Since $E(X_i) = 1/p$ $(i = 1, 2, \ldots, r)$,
$$E(X) =$$

Example 10.8

- Let *X* be a hypergeometric random variable with probability mass function

$$p(x) = P(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}},$$

$$n \le \min(D, N-D), \quad \mathrm{x} = 0,1,2,\ldots, n$$

- Then X is the number of defective items among n items drawn at random and without replacement from an urn containing D defective and N-D nodefective items.

Example 10.8

To calculate E(X),

Let A_i be the event that the ith item drawn is defective.

Also, for $i = 1,2,\ldots,n$, let $X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

Hence, $E(X) = E(X_1) + E(X_2) + \ldots + E(X_n)$

where for $i = 1,2,\ldots,n$,

$E(X_i) =$

Therefore, $\mathrm{E(X)} = \dfrac{nD}{N}$