CS 333202: Probability and Statistics
HW4 Part I

1. Let X_i denote the number on the ith ball drawn, $i = 1, \ldots, m$. Then
   $P\{X \leq k\}$
   $= P\{X_1 \leq k, X_2 \leq k, \ldots, X_m \leq k\}$
   $= P\{X_1 \leq k\}P\{X_2 \leq k\} \cdots P\{X_m \leq k\}$
   $= (\frac{k}{n})^m$
   Therefore,
   $P\{X = k\} = P\{X \leq k\} - P\{X \leq k-1\} = (\frac{k}{n})^m - (\frac{k-1}{n})^m$

2. If T is the number of tests needed for a group of 10 people, then

   $$E[T] = (0.9)^{10} + 11[1 - (0.9)^{10}] = 11 - 10(0.9)^{10}$$

3. Let N denote the number of games played.

   (a) $E[N] = 2[p^2 + (1-p)^2] + 3[2p(1-p)] = 2 + 2p(1-p)$
   Differentiation yields that

   $$\frac{d}{dp}E[N] = 2 - 4p$$

   and so the minimum occurs when $2 - 4p = 0$ or $p = 1/2$

   (b) $E[N] = 3[p^3 + (1-p)^3] + 4[3p^2(1-p)p + 3p(1-p)^2(1-p)] + 5[6p^2(1-p)^2] = 6p^4 - 12p^3 + 3p^2 + 3p + 3$
   Differentiation yields

   $$\frac{d}{dp}E[N] = 24p^3 - 36p^2 + 6p + 3$$

   Its value at p=1/2 is easily seen to be 0.

4. (a) $\frac{1}{10}(1 + 2 + \ldots + 10) = \frac{11}{2}$

(b) After 2 questions, the are 3 remaining possibilities with probability 3/5 and 2 with probability 2/5. Hence,

$$E[\text{Number}] = \tfrac{2}{5}(3) + \tfrac{3}{5}[2 + \tfrac{1}{3} + 2(\tfrac{2}{3})] = \tfrac{17}{5}.$$

The above assumes that when 3 remain, you choose 1 of the 3 and ask if that is the one.