CS 333202: Probability and Statistics
HW13 Part II

1. Let I_i equal to 1 if matching on card i and 0 otherwise. Therefore, $E[I_i] = 1/13$

$$E[\text{number of matches}] = E[\sum_{i=1}^{52} I_i] = 52(1/13) = 4$$

2. Let X_i equal to 1 if a changeover occurs on the ith flip and 0 otherwise. Then

$$E[X_i] = P\{i-1 \text{ is H}, i \text{ is T}\} + P\{i-1 \text{ is T}, i \text{ is H}\} = 2(1-p)p$$
$$E[\text{number of changeovers}] = E[\sum X_i] = \sum_1^n E[X_i] = 2p(n-1)(1-p)$$

3. (a) It is the mean of a geometric random variable. Therefore,

$$E[N] = E[\text{number until first type 1}] - 1 = \frac{1}{P_1} - 1$$

(b) Let

$$X_j = \begin{cases} 1 & \text{a type } j \text{ is caught before type 1} \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned} E[\sum_{j\neq 1} X_j] &= \sum_{j\neq 1} E[X_j] \\ &= \sum_{j\neq 1} P\{\text{type } j \text{ before type } i\} \\ &= \sum_{j\neq 1} P_j/(P_1+P_j) \end{aligned}$$

where the last equality follows upon conditioning on the first time either a type 1 or type j is caught to give.

$$P\{\text{type } j \text{ before type } i\} = P\{j | j \textit{ or } 1\} = P_j/(P_1+P_j)$$

4. Let $Y = log(X)$. Since Y is normal with mean μ and variance σ^2 it follows that its moment generating function is

$$M(t) = E[e^{tY}] = e^{\mu t + \sigma^2 t^2/2}$$

Hence, since $X = e^Y$, we have that $E[X] = M(1) = e^{\mu + \sigma^2/2}$ and $E[X^2] = M(2) = e^{2\mu + 2\sigma^2}$.
Therefore, $Var(X) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$

5.

$$M_{2X+1}(t) = E[e^{(2X+1)t}] = e^t E(e^{2tX}) = e^t M_X(2t) = \frac{e^t}{1-2t}, t < 1/2$$

6.

$$M_X(t) = \frac{1}{3^4}(e^t + 2)^4 = [\frac{1}{3}e^t + (1 - \frac{1}{3})]^4$$

$\therefore X$ is binomial with parameters 4 and $\frac{1}{3}$

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \left(\begin{array}{c} 4 \\ 0 \end{array} \right) (\frac{1}{3})^0 (\frac{2}{3})^4 + \left(\begin{array}{c} 4 \\ 1 \end{array} \right) (\frac{1}{3})^1 (\frac{2}{3})^3 + \left(\begin{array}{c} 4 \\ 2 \end{array} \right) (\frac{1}{3})^2 (\frac{2}{3})^2 \\ &= \frac{8}{9} \end{aligned}$$