CS 333202: Probability and Statistics
HW13 Part I

1. For $i = 1, 2, \ldots, n$, let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th letter is addressed correctly} \\ 0 & \text{otherwise.} \end{cases}$$

   Clearly,

$$E(X_i) = 1 \cdot \tfrac{1}{n} + 0 \cdot (1 - \tfrac{1}{n}) = \tfrac{1}{n}$$

   Thus, $E(X_1 + X_2 + \cdots + X_n) = n \cdot \frac{1}{n} = 1$ is the desired quantity.

2. Let E_1 be the event that the first three outcomes are heads and the fourth outcome is tails. For $2 \leq i \leq n-3$, let E_i be as defined in the hint. Let E_{n-2} be the event that the outcome $(n-3)$ is tails and the last three outcomes are heads. The expected number of exactly three consecutive heads is
   $E(X_1 + \sum_{i=2}^{n-3} X_i + X_{n-2})$
   $= E(X_1) + \sum_{i=2}^{n-3} E(X_i) + E(X_{n-2})$
   $= P(E_1) + \sum_{i=2}^{n-3} P(E_i) + P(E_{n-2})$
   $= (\frac{1}{2})^4 + \sum_{i=2}^{n-3} (\frac{1}{2})^5 + (\frac{1}{2})^4$
   $= (\frac{1}{2})^3 + (n-4)(\frac{1}{2})^5$
   $= \frac{n}{32}$.

3. $R(1)$, the number of runs of 1, can be expressed as

$$R(1) = \textstyle\sum_{i=1}^{n+m} I_i$$

   and thus

$$E[R(1)] = \sum_{i=1}^{n+m} E[I_i]$$

Now,

$$E[I_1] = P(\text{“1” in position 1}) = \frac{n}{n+m}$$

and for $1 < i \leq n+m$,

$$E[I_i] = P(\text{“0” in position } i-1, \text{ “1”” in position } i) = \frac{m}{n+m}\frac{n}{n+m-1}$$

Hence

$$E[R(1)] = \frac{n}{n+m} + (n+m-1)\frac{nm}{(n+m)(n+m-1)}$$

Similarly, $E[R(0)]$, the expected number of runs of 0's, is

$$E[R(0)] = \frac{m}{n+m} + \frac{nm}{n+m}$$

and the expected number of runs of either type is

$$E[R(1) + R(0)] = 1 + \frac{2nm}{n+m}$$