CS 333202: Probability and Statistics
HW12 Part II

1. $M_X(t) = E(e^{tX}) = \sum_{x=1}^{5} e^{tx}p(x) = \frac{1}{5}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t})$.

2. (a) By definition,
   $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$

   (b) From
   $$M_X'(t) = \lambda e^t e^{\lambda(e^t-1)}$$
   and
   $$M_X''(t) = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}$$
   we obtain $E(X) = M_X'(0) = \lambda$ and $E(X^2) = M_X''(0) = \lambda^2 + \lambda$. Therefore,
   $$\mathrm{Var}(X) = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

   (c) We see from (a) that $M(t) = e^{3(e^t-1)}$ is the moment generating function of a Poisson random variable with mean 3. Hence, by the one-to-one correspondence between moment generating functions and distribution functions, it follows that X must be a Poisson random variable with mean 3. Thus $P(X = 0) = e^{-3}$.

3. Note that

   $M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}\cdot 2(\frac{1}{3})^x = 2\sum_{x=1}^{\infty} e^{tx}\cdot e^{-x\ln 3} = 2\sum_{x=1}^{\infty} e^{x(t-\ln 3)}$

   Restricting the domain of $M_X(t)$ to the set $\{t : t < \ln 3\}$ and using the geometric series theorem, we get

$$M_X(t) = 2(\tfrac{e^{t-\ln 3}}{1-e^{t-\ln 3}}) = \tfrac{2e^t}{3-e^t}$$

(Note that $e^{-\ln 3}$.) Differentiating $M_X(t)$, we obtain

$$M_X'(t) = \tfrac{6e^t}{(3-e^t)^2},$$

which gives $E(X) = M_X'(0) = 3/2$.