CS 333202: Probability and Statistics
HW12 Part I

1. Let $p(x, y)$ be the joint probability mass function of X and Y . Clearly,

$$p_Y(5) = (\tfrac{12}{13})^4(\tfrac{1}{13})$$

and

$$p(x,5) = \begin{cases} (\frac{11}{13})^{x-1}(\frac{1}{13})(\frac{12}{13})^{4-x}(\frac{1}{13}) & x < 5 \\ 0 & x = 5 \\ (\frac{11}{13})^4(\frac{1}{13})(\frac{12}{13})^{x-6}(\frac{1}{13}) & x > 5 \end{cases}$$

Using these, we have that
$E(X \mid Y = 5) = \sum_{x=1}^{\infty} x p_{X|Y}(x|5) = \sum_{x=1}^{\infty} x \frac{p(x,5)}{p_Y(5)}$
$= \sum_{x=1}^{4} \frac{1}{11} x (\frac{11}{12})^x + \sum_{x=6}^{\infty} x (\frac{11}{12})^4 (\frac{1}{13}) (\frac{12}{13})^{x-6}$
$= 0.72932 + (\frac{11}{12})^4(\frac{1}{13}) \sum_{y=0}^{\infty} (y+6)(\frac{12}{13})^y$
$= 0.72932 + (\frac{11}{12})^4(\frac{1}{13}) [\sum_{y=0}^{\infty} y (\frac{12}{13})^y + 6 \sum_{y=0}^{\infty} (\frac{12}{13})^y]$
$= 0.72932 + (\frac{11}{12})^4(\frac{1}{13}) [\frac{12/13}{(1/13)^2} + 6 \frac{1}{1-(12/13)}] = 13.412.$

Remark: In successive draws of cards from an ordinary deck of 52 cards, one at a time, randomly, and with replacement, the expected value of the number of draws until the first ace is $1/(1/13) = 13$. This exercise shows that knowing the first king occurred on the fifth trial will increase, on the average, the number of trials until the first ace 0.412 draws.

2. Let Y be the total number of heads obtained. Let X be the total number of heads in the first 10 flips. For $2 \le x \le 10$,

$$p_{X|Y}(x|12) = \frac{p(x,12)}{p_Y(12)} = \frac{\binom{10}{x}(\frac{1}{2})^{10} \cdot \binom{10}{12-x}(\frac{1}{2})^{10}}{\binom{20}{12}(\frac{1}{2})^{20}} = \frac{\binom{10}{x} \cdot \binom{10}{12-x}}{\binom{20}{12}}$$

This is the probability mass function of a hypergeometric random variable with parameters $N = 20$, $D = 10$, and $n = 12$. Its expected value is $\frac{nD}{N} = \frac{12\times 10}{20} = 6$, as expected.

3. The problem is equivalent to the following: Two points X and Y are selected independently and at random from the interval $(0, \ell)$. What is the probability that the length of at least one interval is less than $\ell/20$? The solution to this problem is as follows:
$P(\min(X,\, Y - X,\, \ell - Y) < \frac{\ell}{20} \mid X < Y)P(X < Y) + P(\min(Y,\, X - Y,\, \ell - X) < \frac{\ell}{20} \mid X > Y)P(X > Y)$
$= 2P(\min(X,\, Y - X,\, \ell - Y) < \frac{\ell}{20} \mid X < Y)P(X < Y)$
$= 2P(\min(X,\, Y - X,\, \ell - Y) < \frac{\ell}{20} \mid X < Y) \cdot \frac{1}{2}$
$= 1 - P(\min(X,\, Y - X,\, \ell - Y) \geq \frac{\ell}{20} \mid X < Y)$
$= 1 - P(X \geq \frac{\ell}{20},\, Y - X \geq \frac{\ell}{20},\, \ell - Y \geq \frac{\ell}{20} \mid X < Y)$
$= 1 - P(X \geq \frac{\ell}{20},\, Y - X \geq \frac{\ell}{20},\, Y \leq \frac{19\ell}{20} \mid X < Y)$.
Now $P(X \geq \frac{\ell}{20},\, Y - X \geq \frac{\ell}{20},\, Y \leq \frac{19\ell}{20} \mid X < Y)$ is the area of the region

$$\{(x, y) \in \mathbf{R^2} : 0 < x < \ell,\, 0 < y < \ell,\, x \geq \tfrac{\ell}{20},\, y - x \geq \tfrac{\ell}{20},\, y \leq \tfrac{19\ell}{20}\}$$

divided by the area of the triangle

$$\{(x, y) \in \mathbf{R^2} : 0 < x < \ell,\, 0 < y < \ell,\, y > x\};$$

that is,

$$\frac{\frac{17\ell}{20} \times \frac{17\ell}{20}}{2} \div \frac{\ell^2}{2} = 0.7225.$$

Therefore, the desired probability is $1 - 0.7225 = 0.2775$