CS 333202: Probability and Statistics
HW11 Part I

1. If we let X and Y denote, respectively, the time past 12 that the man and the woman arrive, then X and Y are independent random variable, each of which is uniformly distributed over $(0, 60)$. The desired probability, $P(X + 10 < Y) + P(Y + 10 < X)$, which by symmetry equals $2P(X + 10 < Y)$, is obtained as follows:
   $2P(X + 10 < Y)$
   $= 2\int\int_{x+10<y} f(x, y)dxdy$
   $= 2\int\int_{x+10<y} f_X(x)f_Y(y)dxdy$
   $= 2\int_{10}^{60}\int_0^{y-10}(\frac{1}{60})^2 dxdy$
   $= (\frac{2}{60})^2\int_{10}^{60}(y - 10)dy$
   $= \frac{25}{36}$

2. Since

   $$f_{X,Y,Z}(x, y, z) = f_X(x)f_Y(y)f_Z(z) \qquad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$$

   we have
   $P(X \ge YZ)$
   $= \int\int\int_{x\ge yz} f_{X,Y,Z}(x, y, z)dxdydz$
   $= \int_0^1\int_0^1\int_{yz}^1 dxdydz$
   $= \int_0^1\int_0^1(1 - yz)dydz$
   $= \int_0^1(1 - \frac{z}{2})dz$
   $= \frac{3}{4}$

3. Let G and g be the probability distribution and probability density functions of $\max(X, Y)/\min(X, Y)$. Then $G(t) = 0$ if $t < 1$. For $t \le 1$,
   $G(t)$

$= P(\frac{\max(X,Y)}{\min(X,Y)} \leq t)$
$= P(\max(X,Y) \leq t \min(X,Y))$
$= P(X \leq t \min(X,Y), Y \leq t \min(X,Y))$
$= P(\min(X,Y) \geq \frac{X}{t}, \min(X,Y) \geq \frac{Y}{t}))$
$= P(X \geq \frac{X}{t}, Y \geq \frac{X}{t}, X \geq \frac{Y}{t}, Y \geq \frac{Y}{t}) = P(Y \geq \frac{X}{t}, X \geq \frac{Y}{t})$
$= P(\frac{X}{t} \leq Y \leq tX)$.
This quantity is the area of the region

$$\{(x,y) : 0 < x < 1,\, 0 < y < 1,\, \frac{x}{t} \leq y \leq tx\}$$

which is equal to $(t-1)/t$. Hence

$$G(t) = \begin{cases} 0 & t < 1 \\ \frac{t-1}{t} & t \geq 1 \end{cases}$$

and therefore,

$$g(t) = G'(t) = \begin{cases} \frac{1}{t^2} & t \geq 1 \\ 0 & \text{elsewhere} \end{cases}$$